

Combinatorics of (perturbative) Quantum Field Theory*

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September 2000

Abstract

We review the structures imposed on perturbative QFT by the fact that its Feynman diagrams provide Hopf and Lie algebras. We emphasize the role which the Hopf algebra plays in renormalization by providing the forest formulas. We exhibit how the associated Lie algebra originates from an operadic operation of graph insertions. Particular emphasis is given to the connection with the Riemann–Hilbert problem. Finally, we outline how these structures relate to the numbers which we see in Feynman diagrams.

*hep-th/0010059, MZ-TH/00-42

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1 Introduction

Renormalization (see [1] for a classical textbook treatment) has been settled as a self-consistent approach to the treatment of short-distance singularities in the perturbative expansion of quantum field theories thanks to the work of Bogoliubov, Parasiuk, Hepp, Zimmermann, and followers. Nevertheless, its intricate combinatorics went unrecognized for a long time. In this review we want to describe the results in a recent series of papers devoted to the Hopf algebra structure of quantum field theory (QFT) [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. These results were obtained during the last three years, starting from first papers on the subject [2, 3, 4] and flourishing in intense collaborations with Alain Connes [5, 6, 7, 8, 9] and David Broadhurst [10, 11, 12].

We will review the results obtained so far in a fairly informative style, emphasizing the underlying ideas and concepts. Technical details and mathematical rigor can be found in the above-cited papers, while it is our present purpose to familiarize the reader with the key ideas. Furthermore, we intend to spell out lines for further investigation, as it more and more becomes clear that this Hopf algebra structure provides a very fine tool for a better understanding of a correct mathematical formulation of QFT as well as for applications in particle and statistical physics.

Nevertheless, we will use one concept for the first time in this paper: we will introduce an operad of Feynman graphs, as it is underlying many of the operations involved in the Hopf and Lie algebras built on Feynman graphs.

2 The Hopf algebra structure: trees and graphs

Let us start right away with the consideration of how rooted trees and Feynman graphs are connected in perturbative QFT.

2.1 Basic considerations

There are two basic operations on Feynman graphs which govern their combinatorial structure as well as the process of renormalization. The question to what extent they also determine analytic properties of Feynman graphs is one of these future lines of investigations, with first results in [13, 14]. We will comment in some detail on this aspect later on.

These two basic operations are the disentanglement of a graph into subgraphs, and the opposite operation, plugging a subgraph into another one. Let us consider the disentanglement of a graph first.

We consider the following three-loop vertex-correction ,

$$\Gamma = \text{triangle diagram with a loop} .$$

We regard it as a contribution to the perturbative expansion of ϕ^3 theory in six spacetime dimensions, where this theory is renormalizable.¹ , contains one interesting subgraph, the one-loop self-energy graph

$$\gamma = \text{---} \bigcirc \text{---} .$$

We are interested in it because it is the only subgraph which provides a divergence, and the whole UV-singular structure comes from this subdivergence and from the overall divergence of , itself. Let

$$, \, _0 := , \, / \gamma = \text{---} \bigtriangleup \text{---}$$

be the graph where we shrink γ to a point. From the analytic expressions corresponding to , , to , $_0$ and to γ we can form the analytic expression corresponding to the renormalization of the graph , . It is given by

$$, \, - R(,) - R(\gamma), \, _0 + R(R(\gamma), \, _0), \quad (1)$$

where we still abuse, in these introductory remarks, notation by using the same symbol , for the graph and the analytic expression corresponding to it. We do so as we want to emphasize for the moment that the crucial step in obtaining this expression is the use of the graph , and its disentangled pieces, γ and , $_0 = , \, / \gamma$. The analytic expressions will come as characters on these Hopf algebra elements, and we will discuss these characters in detail below. Diagrammatically, the above expression reads

$$\text{---} \bigtriangleup \text{---} = R(\text{---} \bigtriangleup \text{---}) - R(\text{---} \bigcirc \text{---}) \text{---} \bigtriangleup \text{---} + R(R(\text{---} \bigcirc \text{---}) \text{---} \bigtriangleup \text{---}).$$

The unavoidable arbitrariness in the so-obtained expression lies in the choice of the map R which we suppose to be such that it does not modify the short-distance singularities (UV divergences) in the analytic expressions corresponding to the graphs. This then renders the above combination of four terms finite. If there were no subgraphs, a simple subtraction , $-R(,)$ would suffice to eliminate the short-distance singularities, but the necessity to obtain local counterterms forces us to first subtract subdivergences, which is achieved by Bogoliubov's famous \bar{R} operation [1], which delivers here:

$$, \, \rightarrow \bar{R}(,) = , \, - R(\gamma), \, _0. \quad (2)$$

¹External lines are amputated, but still drawn, in a convenient abuse of notation. In the massless case considered here no further notation is needed for insertions into propagators. In the general case (massive theories, spin) the external structures defined in [5] are a sufficient tool.

This provides two of the four terms above. Amongst them, these two are free of subdivergences and hence provide only a local overall divergence. The projection of these two terms into the range of R provides the other two terms, which combine to the counterterm

$$Z_\Gamma = -R(\Gamma) + R(R(\Gamma), \Gamma_0) \quad (3)$$

of Γ , Γ_0 and subtracting them delivers the finite result above by the fact that the UV divergences are not changed by the renormalization map R .²

The basic operation here is the disentanglement of the graph Γ into pieces γ and Γ/γ , and this very disentanglement gives rise to a Hopf algebra structure, as was first observed in [2]. This Hopf algebra has a role model: the Hopf algebra of rooted trees. We first want to get an idea about this universal Hopf algebra after which all the Hopf algebras of Feynman graphs are modeled.

Consider the two graphs

$$\Gamma_1 = \text{triangle with internal line}, \quad \Gamma_2 = \text{triangle with internal line}.$$

They have one common property: both of them can be regarded as the graph

$$\Gamma_0 = \Gamma_1/\gamma = \Gamma_2/\gamma = \text{triangle with internal line}$$

into which the subgraph

$$\gamma = \text{circle}$$

is inserted, at two different places though. But as far as their UV-divergent sectors go they both realize a rooted tree of the form given in Fig.(1), in the language of [2] both graphs Γ_1, Γ_2 correspond to a parenthesized word of the form

$$((\text{circle}) \text{triangle}).$$

In [2] such graphs were considered to be equivalent, as the combinatorial process of renormalization produces exactly the same terms for both of them. We will formulate this equivalence in a later section using the language of operads.

The combinatorics of renormalization is essentially governed by this bookkeeping process of the hierarchies of subdivergences, and this bookkeeping is what is delivered by rooted trees. They are just the appropriate tool to store the hierarchy of disjoint and nested subdivergences. Another example given in Fig.(2) might be better suited than any formalism to make this clear.

²Locality is connected to the absence of subdivergences: if a graph has a sole overall divergence, UV singularities only appear when all loop momenta tend to infinity jointly. Regarding the analytic expressions corresponding to a graph as a Taylor series in external parameters like masses or momenta, powercounting establishes that only the coefficients of the first few polynomials in these parameters are UV singular. Hence they can be subtracted by a counterterm polynomial in fields and their derivatives. The argument fails as long as one has not eliminated all subdivergences: their presence can force each term in the Taylor series to be divergent.

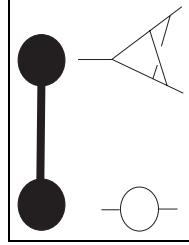


Figure 1: A decorated rooted tree with two vertices, each decorated by a graph without subdivergences (assuming this is an example in ϕ^3 theory in six dimensions). The root (by our convention the uppermost vertex) is decorated by the graph $\gamma_0 = \gamma_1/\gamma = \gamma_2/\gamma$ which we obtain when we shrink the subdivergence γ to a point in either γ_1 or γ_2 . The vertex decorated by the one-loop self-energy γ corresponds to this subdivergence, and the rooted tree stores the information that this divergence is nested in the other graph. The information at which place the subdivergence is to be inserted is not stored in this notation. The hierarchy which determines the recursive mechanism of renormalization is independent of this information. It can easily be restored allowing marked graphs as decorations, or one could directly formulate the Hopf algebra on graphs as we do below.

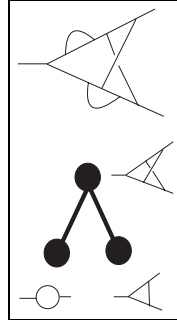


Figure 2: This graph has a hierarchy of divergences given by two disjoint subdivergences, the self-energy γ and a one-loop vertex-correction $\tilde{\gamma}$, so that its divergent structure represents the decorated rooted tree indicated. As a parenthesized word, the graph corresponds to $((\gamma)(\tilde{\gamma}), \gamma_0)$. There are, by the way, $5 \times 6 = 30$ graphs which are all equivalent in the sense that they represent this rooted tree or parenthesized word, generated by the 5 internal vertices and 6 internal edges which provide places for insertion in γ_0 .

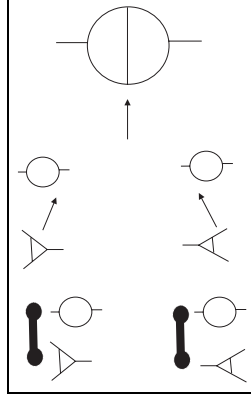


Figure 3: Finding the two ways of getting the overlapping graph Ω . There are two vertices in the one-loop self-energy into which the one-loop vertex correction can be inserted. Both result in the same graph. The short-distance singularities in Ω arise from two sectors, described by two decorated rooted trees.

At this stage, the reader should wonder what to make out of graphs which have overlapping divergences. This can be best understood when we turn to the other basic operation on graphs: plugging them into each other. On the one hand, for the non-overlapping graphs γ_1, γ_2 above there is a unique way to obtain them from

$$\gamma_0 = \gamma_1 / \gamma = \gamma_2 / \gamma = \text{triangle}$$

and the self-energy γ . We plug γ into the vertex-correction at an appropriate internal line to obtain these graphs. This operation will be considered in some more detail in a later section. On the other hand, for a graph which contains overlapping divergences we have typically no unique manner, but several ways instead, how to obtain it. For example,

$$\Omega = \text{circle with vertical line}$$

can be obtained in the two ways indicated in Fig.(3).

Each of these ways corresponds to a rooted tree [4], and the sum over all these rooted trees bookkeeps the subdivergent structures of a graph with overlapping divergences correctly. The resolution of overlapping divergences into rooted trees corresponds to the determination of Hepp sectors, and amounts to a resolution of overlapping subsets into nested and disjoint subsets generally [4].³

³The remarks above are specific to theories which have trivalent couplings. In general, the determination of divergent sectors still leads to rooted trees [4]. A concrete example how the Hopf algebra structure appears in ϕ^4 -theory can be found in [20]. Also, resolving the

One remark is in order: the very fact that overlapping divergences can be reduced to divergences which have a tree-hierarchy has a deeper reason: the short-distance singularities of QFT result from confronting products of distributions which are well-defined on the configuration space of vertices located at distinct space-time points, but which become ill-defined along diagonals [17, 18]. But then, the various possible ways how an ensemble of distinct points can collapse to (sub-)diagonals is known to be stratified by rooted trees [19], and this is what essentially ensures that the Hopf algebra structure of these trees can reproduce the forest formulas of perturbative QFT. Let us then have a closer look at the connection between graphs and rooted trees.

2.2 Sector decomposition and rooted trees

Consider the Feynman graph Ω once more, as given in Fig.(4). It corresponds to a contribution to the perturbative expansion in the coupling constant g of the theory to order g^4 . It has short-distance (UV) singularities which are apparent in the following sectors

$$I_1 := \{1, 2, 3\}, I_2 := \{2, 3, 4\}, I = \{1, 2, 3, 4\},$$

which give the label of the vertices participating in the divergent (sub-)graphs. Note that the sectors overlap: $I_1 \cap I_2 \neq \emptyset$. The singularities are stratified so that they can be represented as rooted trees, as described in Fig.(4). In this stratification of sectors each node at the rooted tree corresponds to a Feynman graph which connects the vertices attached to the node by propagators in a manner such that it has no subdivergences. We call such graphs primitive graphs. Each primitive graph is only overall divergent.

Now, where do singularities reside? Typically, if we write down analytic expressions in terms of momentum integrals, UV-divergences appear when the loop momenta involved in a primitive graph tend to infinity jointly, and this can be detected by powercounting over edges and vertices in the graph. On the other hand, we can consider Feynman rules in coordinate space. Then, the UV-singular integrations over momenta become short distance singularities. Again, they creep in from the very fact that closed loops, cycles in the graph, force the integration over the positions of vertices to produce ill-defined products of distributions with coinciding support. Powercounting amounts to a check of the scaling degree of the relevant distributions and ultimately determines the appearance of a short distance singularity at the diagonal under consideration.

The short distance singularities of Feynman graphs then come solely from regions where all vertices are located at coinciding points. One has no problem to define the Feynman integrand in the configuration space of vertices at distinct locations, while a proper extension to diagonals is what is required.

overlapping divergences in terms of decorated rooted trees determines the appropriate set of primitive elements of the Hopf algebra, which can for example be systematically achieved by making use of Dyson–Schwinger equations [2, 15], see also [21].

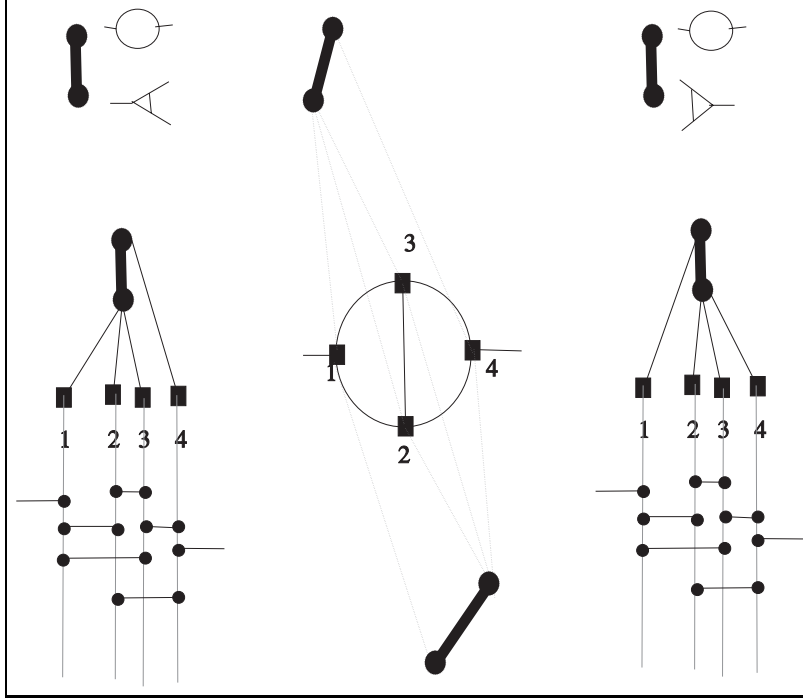


Figure 4: A different way of looking at the graph Ω . We label its vertices by $1, \dots, 4$. Then, the set of vertices $\{1, 2, 3\}$ belongs to a vertex subgraph, as does the set $\{2, 3, 4\}$. The fact that both are proper subsets of the set of all vertices $\{1, 2, 3, 4\}$ is again reflected in a tree-like hierarchy. A short distance singularity appears when these labelled vertices of a divergent graph collapse to a point. This point (a diagonal) constitutes one vertex of a rooted tree, with the root corresponding to the collapse of all vertices to the overall diagonal jointly. Again, the graph gives rise to two rooted trees, which corresponds to the two divergent sectors along two different diagonals. When we blow up the vertices $\{1, 2, 3, 4\}$ of the graph to vertical lines, we can represent the edges of the Feynman graph as horizontal chords, and we regain the graph by shrinking the vertical lines to a point. The Hopf algebra structure operates on the bold black rooted trees, as they store the information which diagonals contain short-distance singularities in the graph under consideration.

In the above, the two divergent subgraphs are ill-defined along the diagonals $x_1 = x_2 = x_3$ and $x_2 = x_3 = x_4$ while the overall divergence corresponds to the main diagonal $x_1 = x_2 = x_3 = x_4$.

Due to the Hopf algebra structure of Feynman graphs we can define the renormalization of all such sectors without making recourse to any specific analytic properties of the expressions (Feynman integrals) representing those sectors. The only assumption we make is that in a sufficiently small neighborhood of such an ultralocal region (the neighborhood of a diagonal) we can define the scaling degree, –the powercounting–, in a sensible manner.

Apart from this assumption our approach is purely combinatorial and in particular independent of the geometry of the underlying spacetime manifold.

Fig.(4) also gives a first idea why the Hopf algebra of undecorated rooted trees is the universal object underlying the Hopf algebras of Feynman graphs. The essential combinatorics needed to obtain local counterterms will solely use cuts on these rooted trees which are drawn in bold black lines in the figure, with no further operation on decorations. Different theories just differ by having different types of chords and vertices, while to each chord and vertex in the figure we assign the appropriate scaling degree, the weight with which they contribute to the powercounting.

One further remark is in order: the existence of a purely combinatorial solution coincides with the result of Brunetti and Fredenhagen [22], who showed that the renormalization mechanism is indeed unchanged in the context of curved manifolds in a detailed local analysis using the Epstein–Glaser mechanism. To my mind, quite generally, the Hopf algebra can be used to make sense out of extensions of products of distributions to diagonals of configuration spaces even before we decide by which class of (generalized) functions we want to realize these extensions. While consistency of the Hopf algebra approach to renormalization with the Epstein–Glaser formalism was settled once the Hopf algebra was directly formulated on graphs [4, 8], it was also addressed at a notational level making use of configuration space Feynman graphs in [20]. Still, one should regard the splitting of distributions itself as the first instance where a representation of the Hopf algebra is realized, so that properties like Lorentz covariance appear as properties of the representation alone, maintaining a proper separation of the combinatorics of the Bogoliubov recursion from the analytic properties of the functions defined over the configuration space, enabling also a direct formulation on the level of time-ordered products instead of Feynman graphs.

Once more, that the Hopf algebra structure coming in is the one of rooted trees should be no surprise: limits to diagonals in configuration spaces are stratified by rooted trees [19], and it is the Hopf algebra structure of these rooted trees which describes the combinatorics of renormalization, as we will see. The Hopf algebra of rooted trees will be the role model for all the Hopf algebras of Feynman graphs for a specifically chosen QFT, a classifying space in technical terms (see Theorem 2, section 3 in [5]), while each such chosen QFT probes

$$\boxed{B_{-}(\text{rooted tree}) = \text{rooted tree}}$$

Figure 5: The action of B_{-} on an undecorated rooted tree.

the short distance singularities according to its Feynman graphs. The resulting iterative procedure gives rise to the Hopf algebra of rooted trees which was first described, in the equivalent language of parenthesized words, in [2] and then in its final notation in [5]. It is now time to describe this Hopf algebra of rooted trees in some detail.

2.3 The Hopf algebra of undecorated rooted trees

We follow section II of [5] closely. A *rooted tree* t is a connected and simply-connected set of oriented edges and vertices such that there is precisely one distinguished vertex which has no incoming edge. This vertex is called the root of t . Further, every edge connects two vertices and the *fertility* $f(v)$ of a vertex v is the number of edges outgoing from v . The trees being simply-connected, each vertex apart from the root has a single incoming edge (we could attach, if we like, an extra edge to the root as well, for a more common treatment). Each vertex in such a rooted tree corresponds to a divergent sector in a Feynman diagram. The rooted trees store the hierarchy of such sectors. We will always draw the root as the uppermost vertex in figures, and agree that all edges are oriented away from the root.

As in [5], we consider the (commutative) algebra of polynomials over \mathbb{Q} in rooted trees, where the multiplication $m(t, t')$ of two rooted trees is their disjoint union, so we can draw them next to each other in arbitrary order, and the unit with respect to this multiplication is the empty set.⁴ Note that for any rooted tree t with root r which has fertility $f(r) = n \geq 0$, we have trees t_1, \dots, t_n which are the trees attached to r .

Let B_{-} be the operator which removes the root r from a tree t , as in Fig.(5):

$$B_{-} : t \rightarrow B_{-}(t) = t_1 t_2 \dots t_n. \quad (4)$$

We extend the action of B_{-} to a product of rooted trees by a Leibniz rule, $B_{-}(XY) = B_{-}(X)Y + XB_{-}(Y)$. We also set $B_{-}(t_1) = 1$, $B_{+}(1) = t_1$, where t_1 is the rooted tree corresponding to the root alone.

Let B_{+} be the operation which maps a monomial of n rooted trees to a new rooted tree t which has a root r with fertility n which connects to the n roots

⁴We restrict ourselves to one-particle irreducible diagrams for the moment. Then, the disjoint union of trees corresponds to the disjoint union of graphs. One could also set up the Hopf algebra structure such that one-particle reducible graphs correspond to products of rooted trees [2].

$$\boxed{B_+(\cdot \vdots) = \text{tree diagram}}$$

Figure 6: The action of B_+ on a monomial of trees.

of t_1, \dots, t_n :

$$B_+ : t_1 \dots t_n \rightarrow B_+(t_1 \dots t_n) = t. \quad (5)$$

This is clearly the inverse to the action of B_- on single rooted trees. One has

$$B_+(B_-(t)) = B_-(B_+(t)) = t \quad (6)$$

for any rooted tree t . Fig.(6) gives an example.

All the operations described here have a straightforward generalization to decorated rooted trees, in which case the operator B_+ carries a further label to indicate the decoration of the root [5]. We will not use decorated rooted trees later, as we will directly formulate the Hopf algebras of specific QFTs on Feynman graphs. The Hopf algebra of undecorated rooted trees is the universal object [5] for all those Hopf algebras, and hence we describe it here in some detail.

Note that while $[B_+, B_-](t) = 0$ for any single rooted tree t , this commutator is non-vanishing on products of trees. Obviously, one always has $\text{id} = B_- B_+$, while $B_+ B_-$ acts trivially only on single rooted trees, not on their product.⁵

We will introduce a Hopf algebra on our rooted trees by using the possibility to cut such trees in pieces. For the reader not familiar with Hopf algebras, let us mention a few very elementary facts first. An algebra A is essentially specified by a binary operation $m : A \times A \rightarrow A$ (the product) fulfilling the associativity $m(m(a, b), c) = m(a, m(b, c))$ so that to each two elements of the algebra we can associate a new element in the algebra, and by providing some number field \mathbb{K} imbedded in the algebra via $E : \mathbb{K} \rightarrow A, k \rightarrow k1$. In a coalgebra we do the opposite, we disentangle each algebra element: each element a is decomposed by the coproduct $\Delta : A \rightarrow A \times A$ in a coassociative manner, $(\Delta \times \text{id})\Delta(a) = (\text{id} \times \Delta)\Delta(a)$. Further, the unit 1 of the algebra, $m(1, a) = m(a, 1) = a$, is dualized to the counit \bar{e} in the coalgebra, $(\bar{e} \times \text{id})\Delta(a) = (\text{id} \times \bar{e})\Delta(a) = a$. If the two operations m, Δ are compatible (the coproduct of a product is the product of the coproducts), we have a bialgebra, and if there is a coinverse, the celebrated antipode $S : H \rightarrow H$, as well, we have a Hopf algebra. While in the algebra the

⁵This has far reaching consequences and is closely connected to the fact that logarithmic derivatives (with respect to the log of some scale say) of Z -factors are finite quantities. Indeed, Z -factors can be regarded as formal series over Feynman diagrams graded by the loop number starting with 1, and their logarithm defines a series in graphs which typically demands that commutators like $[B_+, B_-](t_1 t_1)$ are a primitive element in the Hopf algebra, and hence provide only a first order pole [16, 12]. This is a first instance of a t'Hooft relation to which we turn later when we review the results of [9].

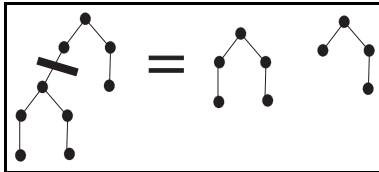


Figure 7: An elementary cut c splits a rooted tree t into two components. We remove the chosen edge and get two components. Both are rooted trees in an obvious manner: one contains the vertex which was the old root and the root of the other is provided by the vertex which was at the endpoint (edges are oriented away from the root) of the removed edge.

unit, the inverse and the product are related by $m(a, a^{-1}) = m(a^{-1}, a) = 1$, the counit, the coproduct and the coinverse are related by $m(S \times \text{id})\Delta = E \circ \bar{e}$. A thorough introduction can be found for example in [23].

To define a coproduct for rooted trees we are hence looking for a map which disentangles rooted trees. We start with the most elementary possibility. An *elementary cut* is a cut of a rooted tree at a single chosen edge, as indicated in Fig.(7). By such a cutting procedure, we will obtain the possibility to define a coproduct, as we can use the resulting pieces on either side of the coproduct. It is this cutting operation which corresponds to the disentanglements of graphs discussed before.

Still before introducing the coproduct we introduce the notion of an *admissible cut*, also called a *simple cut* [5]. It is any assignment of elementary cuts to a rooted tree t such that any path from any vertex of the tree to the root has at most one elementary cut, as in Fig.(8). An admissible cut C maps a tree to a monomial in trees. If the cut C contains n elementary cuts, it induces a map

$$C : t \rightarrow C(t) = \prod_{i=1}^{n+1} t_{j_i}. \quad (7)$$

Note that precisely one of these trees t_{j_i} will contain the root of t . Let us denote this distinguished tree by $R^C(t)$. The monomial which is delivered by the $n - 1$ other factors is denoted by $P^C(t)$. In graphs, $P^C(t)$ corresponds to a set of disjoint subgraphs $\cup_i \gamma_i$ which we shrink to a point and take out of the initial graph, corresponding to t , while $R^C(t)$ corresponds to the remaining graph, $t / (\cup_i \gamma_i)$. Admissibility means that there are no further disentanglements in the set $\cup_i \gamma_i$. Hence, a sum over all such sets provides a sum over all unions of subgraphs, as we will discuss below. Arbitrary non-admissible cuts correspond to the notion of forests in the sense of Zimmermann [2, 5].

Let us now establish the Hopf algebra structure. Following [2, 5] we define the counit and the coproduct. The *counit* $\bar{e}: H \rightarrow \mathbb{Q}$ is simple:

$$\bar{e}(X) = 0$$

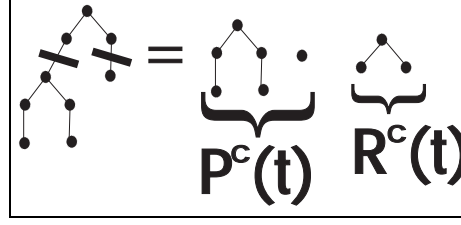


Figure 8: An admissible cut C acting on a tree t . It produces a monomial of trees. One of the factors, $R^C(t)$, contains the root of t .

$$\begin{aligned}
 \Delta(\cdot) &= \cdot \otimes 1 + 1 \otimes \cdot \\
 \Delta(\uparrow) &= \uparrow \otimes 1 + 1 \otimes \uparrow + \cdot \otimes \cdot \\
 \Delta(\uparrow \downarrow) &= \uparrow \downarrow \otimes 1 + 1 \otimes \uparrow \downarrow + \cdot \otimes \uparrow + \uparrow \otimes \cdot \\
 \Delta(\uparrow \downarrow \uparrow) &= \uparrow \downarrow \uparrow \otimes 1 + 1 \otimes \uparrow \downarrow \uparrow + 2 \cdot \otimes \uparrow + \cdot \otimes \uparrow \downarrow \uparrow
 \end{aligned}$$

Figure 9: The coproduct. We work it out for the trees $t_1, t_2, t_{3_1}, t_{3_2}$, from top to bottom.

for any $X \neq 1$,

$$\bar{e}(1) = 1.$$

The *coproduct* Δ is defined by the equations

$$\begin{aligned}
 \Delta(1) &= 1 \otimes 1, \\
 \Delta(t_1 \dots t_n) &= \Delta(t_1) \dots \Delta(t_n), \\
 \Delta(t) &= t \otimes 1 + (id \otimes B_+)[\Delta(B_-(t))],
 \end{aligned} \tag{8}$$

which defines the coproduct on trees with n vertices iteratively through the coproduct on trees with a lesser number of vertices.

The coproduct can be written in a non-recursive manner as [2, 5]

$$\Delta(t) = 1 \otimes t + t \otimes 1 + \sum_{\text{adm. cuts } C \text{ of } t} P^C(t) \otimes R^C(t). \tag{9}$$

Up to now we have established a bialgebra structure. It is actually a Hopf algebra. Following [2, 5] we find the antipode S as

$$S(1) = 1,$$

$$\begin{array}{l}
\mathbf{S}(\bullet) = -\bullet \\
\mathbf{S}(\downarrow) = -\downarrow + \bullet \\
\mathbf{S}(\downarrow \downarrow) = -\downarrow \downarrow + 2 \bullet \downarrow - \dots \\
\mathbf{S}(\downarrow \downarrow \downarrow) = -\downarrow \downarrow \downarrow + 2 \bullet \downarrow \downarrow - \dots
\end{array}$$

Figure 10: The antipode. Again we work it out for the trees $t_1, t_2, t_{3_1}, t_{3_2}$.

$$\begin{aligned}
S(t_1 \dots t_k) &= S(t_1) \dots S(t_k), \\
S(t) &= -t - \sum_{\text{adm. cuts } C \text{ of } t} S[P^C(t)] R^C(t).
\end{aligned} \tag{10}$$

Fig.(10) gives examples for the antipode.

Let us give yet another formula to write the antipode, which one easily derives using induction on the number of vertices [2, 5]:

$$S(t) = - \sum_{\text{all cuts } C \text{ of } t} (-1)^{n_C} P^C(t) R^C(t), \tag{11}$$

where n_C is the number of elementary cuts in C . This time, we have a non-recursive expression, summing over all cuts C , relaxing the restriction to admissible cuts.

By now we have established a Hopf algebra H on rooted trees, using the set of rooted trees, the commutative multiplication m for elements of this set, the unit 1 and counit \bar{e} , the coproduct Δ and antipode S . Still following [2, 5] we allow to label the vertices of rooted trees by Feynman graphs without subdivergences, in the sense described before. Quite general, if Y is a set of primitive elements providing labels, we get a similar Hopf algebra $H(Y)$. The determination of all primitive graphs which can appear as labels corresponds to a skeleton expansion and is discussed in detail in [4]. Instead of using the language of a decorated Hopf algebra we use directly the corresponding Hopf algebra of graphs below.

Let us also mention again that

$$m[(S \otimes \text{id})\Delta(t)] = E \circ \bar{e}(t) \quad (= 0 \text{ for any non-trivial } t \neq 1). \tag{12}$$

As the divergent sectors in Feynman graphs are stratified by rooted trees, we can use the Hopf algebra structure to describe the disentanglement of graphs into

pieces, and it turns out that this delivers the forest formulas of renormalization theory.

Let us now come back to the graph Ω and its representation in Fig.(4). We want to look at the relevant Hopf algebra operations in some detail, which we describe in Fig.(11). The operations described in this figure go through for any QFT whose ultraviolet divergences are local, stratified by rooted trees that is. A renormalizable field theory will only demand a finite number of counterterms in the action, while an effective theory is finite in the number of needed counterterms only for a finite loop order, but the number will actually increase with the loop order. A superrenormalizable theory gives only a truncated representation of rooted trees: higher orders in the perturbative expansion do not deliver new short-distance singularities, and hence the existent divergences are stratified by rooted trees with a restricted number of vertices.

Each short-distance singularity corresponds to a sector which can be described by a rooted tree, which itself notates the hierarchy of singularities. We have a coproduct which describes the job-list [10] of renormalization: we use it to disentangle the singularities located at (sub-)diagonals. The Feynman rules are then providing a character $\phi : H \rightarrow V$ on this Hopf algebra. They map a Hopf algebra element to an analytic expression, typically evaluating in a suitable ring V of Feynman integrands or Laurent polynomials in a regularization parameter. These maps being characters, we have

$$\phi(\gamma_1 \gamma_2) = \phi(\gamma_1) \phi(\gamma_2). \quad (13)$$

Then, renormalization comes from the very simple Hopf algebra property Eq.(12), as we now explain. Let us describe carefully how to use the Hopf algebra structure in the example of Fig.(11). The first thing which we have to introduce, together with our Feynman rules, is a map $R : V \rightarrow V$ which is essentially determined by the choice of a renormalization scheme. The freedom in this choice is essentially what makes up the renormalization group.

The presence of the antipode S allows to consider, for each ϕ , its inverse character $\phi^{-1} = \phi \circ S$. Actually, we have a group structure on characters: to each two characters ϕ, ψ we can assign a new character

$$\phi \star \psi = m_V \circ (\phi \otimes \psi) \circ \Delta,$$

and a unit of the \star -product is provided as

$$\phi \star \eta = \eta \star \phi = \phi$$

and the inverse is indeed provided by the antipode:

$$\phi^{-1} \star \phi = \phi \star \phi^{-1} = \eta,$$

where η comes from the counit and is uniquely defined as $\eta = \psi \circ E \circ \bar{e}$ so that $\eta(1) = 1_V$, $\eta(X) = 0$, $\forall X \neq 1$, and for any arbitrarily chosen character ψ (any character fulfills $\psi(1) = 1_V$).

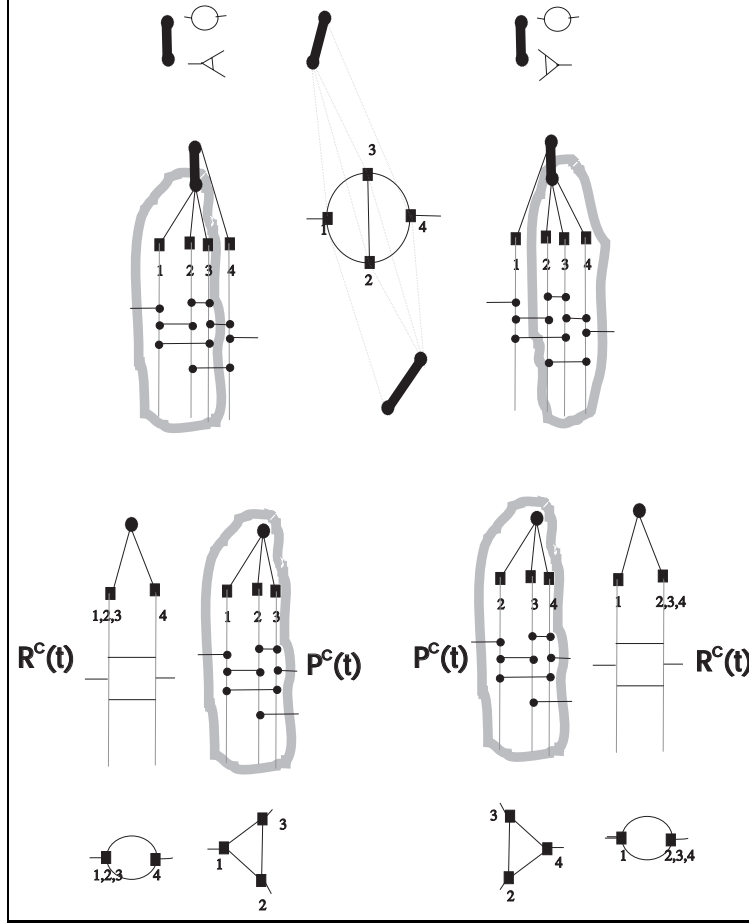


Figure 11: The graph Ω gives rise to two rooted trees corresponding to its two (overlapping) divergent sectors. Each of the two rooted trees allows for a single admissible cut. We implement it in each case by the gray curve which encircles the one vertex which constitutes $P^C(t)$ and the whole chord diagram attached to this subtree. It hence corresponds to a subgraph which is a three-point graph, as three chords are crossed by this gray curve. The cut at the rooted tree then corresponds to shrinking the subgraph to a point, which is a vertex in the remaining graph (a one-loop self-energy). This vertex we have decorated by $\{2, 3, 4\}$ or $\{1, 2, 3\}$. It amounts to a local polynomial insertion in the self-energy. If the vertices so-generated always give rise to polynomial insertions which are part of the action already, we have a renormalizable theory. For a general theory, one will have a variety of different chords represented by different propagators, and a variety of vertices as well. For a renormalizable theory there will be only a finite number of each. It may happen that there are various different vertices into which a graph can shrink, in which case a sum over the corresponding external structures is involved [8].

The next thing to do is to use ϕ and R to define a further character $S_R : H \rightarrow V$ by

$$S_R = -R[\phi(t) + \sum S_R(t')\phi(t'')],$$

where we used the notation $\Delta(t) = t \otimes 1 + 1 \otimes t + \sum t' \otimes t''$. By construction, if we choose $R = \text{id}_V$, the identity map from $V \rightarrow V$, we have $S_{\text{id}_V} = \phi \circ S$.

Now, consider $S_R \star \phi$. We have

$$S_{\text{id}_V} \star \phi = \phi \circ m \circ (S \otimes \text{id}) \circ \Delta = \eta$$

by the Hopf algebra property Eq.(12) above. This guarantees that from regions where R becomes the identity map $\text{id}_V : V \rightarrow V$, we get a vanishing contribution from any non-trivial sector t realized in a Feynman graph, as $\eta(t) = 0$. So if we demand that R leaves short distance singularities unaltered, so that $R = \text{id}_V$ for large loop momenta, we automatically have a vanishing contribution of those singularities to $S_R \star \phi$.⁶

What we see at work here is a general principle of multiplicative subtraction [5]: while for a primitive Hopf algebra element t , $\Delta(t) = t \otimes 1 + 1 \otimes t$, $S_R \star \phi$ amounts simply to the additive operation

$$\phi(t) - R[\phi(t)],$$

for a general Hopf algebra element the coproduct provides a much more refined multiplicative subtraction mechanism, which can obviously be considered for a wide class of Hopf algebras. This principle can certainly be applied in the future not only in the problem of short distance singularities, but in a much wider class of problems, with asymptotic expansions coming to mind immediately.

Fig.(12) describes how the Hopf algebra is realized on the sectors of the graph Ω and how this relates to the Hopf algebra of Feynman graphs to which we now turn.

2.4 The Hopf algebra of graphs

As we already have emphasized the Hopf algebra of rooted trees is the role model for the Hopf algebras of Feynman graphs which underly the process of renormalization when formulated perturbatively at the level of Feynman graphs. The following formulas should be of no surprise after our previous discussions.

First of all, we start considering one-particle irreducible graphs as the linear generators of the Hopf algebra, with their disjoint union as product. We then define a Hopf algebra by a coproduct

$$\Delta(,) = , \otimes 1 + 1 \otimes , + \sum_{\gamma \subset \Gamma} \gamma \otimes , / \gamma, \quad (14)$$

⁶That R leaves short-distance singularities unaltered typically requires that the first few Taylor coefficients in the Feynman integrands, as determined by powercounting, are left unaltered.

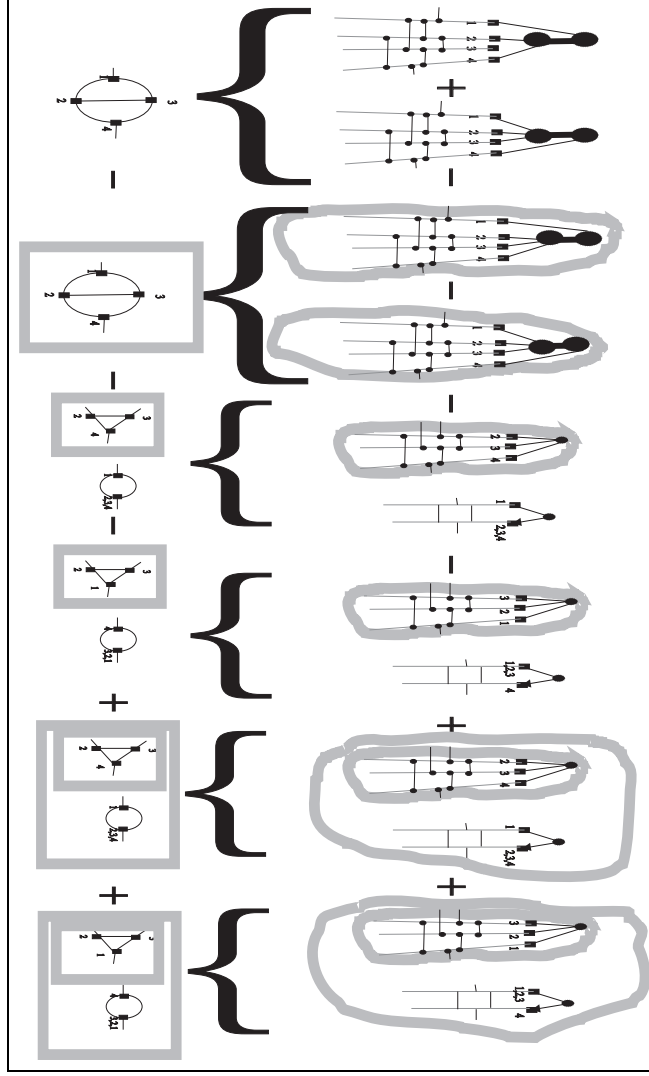


Figure 12: The result of the operation $S_R \star \phi(\Omega)$ graphically, where an application of the operation R is indicated by encircling the graph whose corresponding analytic expression is to be mapped to the range of R by a thick grey line. In the upper row, we see the result in terms of the decorated rooted trees of Fig.(11) while in the second row we see the result directly expressed in terms of Feynman integrals. Again, the map ϕ is not explicitly written out. The grey boxes indicate the full and normal forests of classical renormalization theory [1] and are in one-to-one correspondence with the cuts at the corresponding rooted trees if we incorporate the empty and the full cut in the sum over cuts, so that the two terms $T \otimes 1 + 1 \otimes T$ which appear in any coproduct $\Delta(T) = T \otimes 1 + 1 \otimes T + \sum_{adm. C} P^C(T) \otimes R^C(T)$ can be regarded as generated by the full $(T \otimes 1)$ and the empty cut $(1 \otimes T)$ [5].

where the sum is over all unions of one-particle irreducible (1PI) superficially divergent proper subgraphs and we extend this definition to products of graphs so that we get a bialgebra [8]. The above sum should, when needed, also run over appropriate external structures to specify the appropriate type of local insertion [8] which appear in local counterterms, which we omitted in the above sum for simplicity.⁷

The counit $\bar{\epsilon}$ vanishes, as before, on any non-trivial Hopf algebra element. At this stage we have a commutative, but typically not cocommutative bialgebra. It actually is a Hopf algebra as the antipode in such circumstances comes almost for free as

$$S(\cdot) = -\sum_{\gamma \in \Gamma} S(\gamma) \cdot / \gamma. \quad (15)$$

The next thing we need are Feynman rules, which we regard as maps $\phi : H \rightarrow V$ from the Hopf algebra of graphs H into an appropriate space V .

Over the years, physicists have invented many calculational schemes in perturbative quantum field theory, and hence it is of no surprise that there are many choices for this space. For example, if we want to work on the level of Feynman integrands in a BPHZ scheme, we could take as this space a suitable space of Feynman integrands (realized either in momentum space or configuration space, whatever suits). An alternative scheme would be the study of regularized Feynman integrals, for example the use of dimensional regularization would assign to each graph a Laurent-series with poles of finite order in a variable ε near $\varepsilon = 0$, and we would obtain characters evaluating in this ring. In any case, we will have $\phi(\cdot, \cdot) = \phi(\cdot, \cdot) \phi(\cdot, \cdot)$.

Then, with the calculational scheme chosen and the Feynman rules providing a canonical character ϕ , we will have to make one further choice: a renormalization scheme. This is a map $R : V \rightarrow V$, and we demand that it does not modify the UV-singular structure: in BPHZ language, it should not modify the Taylor expansion of the integrand for the first couple of terms divergent by powercounting. In dimensional regularization, we demand that it does not modify the pole terms in ε .

Finally, the principle of multiplicative subtraction works as before: we define a further character S_R which deforms $\phi \circ S$ slightly and delivers the counterterm for \cdot :

$$S_R(\cdot) = -R[\phi(\cdot)] - R \left[\sum_{\gamma \in \Gamma} S_R(\gamma) \phi(\cdot / \gamma) \right] \quad (16)$$

which should be compared with the undeformed

$$\phi \circ S = -\phi(\cdot) - \sum_{\gamma \in \Gamma} \phi \circ S(\gamma) \phi(\cdot / \gamma). \quad (17)$$

⁷ A simple example exhibited in [8] is the self-energy in massive ϕ^3 theory in six dimensions. It provides two external structures, corresponding to local insertions of counterterms for the $m^2 \phi^2$ and for the $(\partial_\mu \phi)^2$ term.

Then, the classical results of renormalization theory follow suit [2, 4, 5]. We obtain the renormalization of ϕ by the application of a renormalized character

$$\phi \rightarrow S_R \star \phi$$

and the \bar{R} operation as

$$\bar{R}(\phi) = \phi + \sum_{\gamma \in \Gamma} S_R(\gamma) \phi(\cdot/\gamma), \quad (18)$$

so that we have

$$S_R \star \phi = \bar{R}(\phi) + S_R(\phi). \quad (19)$$

In the above, we have given all formulas in their recursive form. Zimmermann's original forest formula solving this recursion is obtained when we trace our considerations back to the fact that the coproduct of rooted trees can be written in non-recursive form, and similarly the antipode. It is not difficult to see that the sum over all cuts corresponds to a sum over all forests, and the notion of full and normal forests of Zimmermann [1] gives rise to appropriate sums over cuts [2, 5], making use of the graphical implementation of cuts as for example in Fig.(12).

3 Rescalings and renormalization schemes

Let us come back to unrenormalized Feynman graphs, and their evaluation by some chosen character ϕ , and let us also choose a renormalization scheme R . The group structure of such characters on the Hopf algebra can be used in an obvious manner to describe the change of renormalization schemes. This has very much the structure of a generalization of Chen's Lemma [3].

3.1 Chen's Lemma

Consider $S_R \star \phi$. Let us change the renormalization scheme from R to R' . How is the renormalized character $S_{R'} \star \phi$ related to the renormalized character $S_R \star \phi$? The answer lies in the group structure of characters:

$$S_{R'} \star \phi = [S_{R'} \star S_R \circ S] \star [S_R \star \phi]. \quad (20)$$

We inserted a unit η with respect to the \star -product in form of $\eta = S_R \circ S \star S_R \equiv S_R^{-1} \star S_R$, and can now read the rerenormalization, switching between the two renormalization schemes, as composition with the renormalized character $S_{R'} \star S_R^{-1}$.⁸

⁸ $S_{R'} \star S_R \circ S$ is a renormalized character indeed: if R, R' are both self-maps of V which do not alter the short-distance singularities as discussed before, then in the ratio $S_{R'} \star S_R \circ S$ those singularities drop out.

Similar considerations apply to a change of scales which determine a character [3]. If ρ is a dimensionful parameter which appears in a character $\phi = \phi(\rho)$,⁹ then the transition $\rho \rightarrow \rho'$ is implemented in the group by acting on the right with the renormalized character $\psi_{\rho, \rho'}^\phi := \phi(\rho) \circ S \star \phi(\rho')$ on $\phi(\rho)$,

$$\phi(\rho') = \phi(\rho) \star \psi_{\rho, \rho'}^\phi. \quad (21)$$

Let us note that this Hopf algebra structure can be efficiently automated as an algorithm for practical calculations exhibiting the full power of this combinatorics [10].

Now, assume we compute Feynman graphs by some Feynman rules in a given theory and decide to subtract UV singularities at a chosen renormalization point μ . This amounts, in our language, to saying that the map S_R is parametrized by this renormalization point: $S_R = S_R(\mu)$. Then, let $\Phi(\mu, \rho)$ be the ratio $\Phi(\mu, \rho) = S_R(\mu) \star \phi(\rho)$. We then have the groupoid law generalizing the before-mentioned Chen's lemma [3]

$$\Phi(\mu, \eta) \star \Phi(\eta, \rho) = \Phi(\mu, \rho). \quad (22)$$

While this looks like a groupoid law, the product of two unrelated ratios $\Phi(\mu_1, \mu_2) \star \Phi(\mu_3, \mu_4)$, as any other product of characters, is always well-defined in the group of characters of the Hopf algebra.

3.2 Automorphisms of the Hopf algebra

In the set-up discussed so far, the combinatorics of renormalization was attributed to a Hopf algebra, while characters of this Hopf algebra took care of the specific Feynman rules and chosen renormalization schemes. Renormalized quantities appear as the ratio of two characters, while divergences drop out in this ratio $S_R \star \phi$.

Typically, such characters introduce a renormalization scale (cut-off, the 't Hooft mass μ in dimensional regularization), and we can use these parameters to describe the change of schemes in a fairly unified manner, as discussed in [3].

These considerations of changes of renormalization schemes are related to another interesting aspect discussed in [3]. So far, we regarded the map R as a self-map in a certain space V . We will not have $R(XY) = R(X)R(Y)$ (for example, minimal subtraction cannot possibly fulfill that the poleterms of a product is the product of the poleterms), but R obeys the multiplicativity constraints

$$R(XY) + R(X)R(Y) = R(XR(Y)) + R(R(X)Y), \quad (23)$$

which ensure that $S_R(\cdot, \cdot, \cdot) = S_R(\cdot, \cdot)S_R(\cdot, \cdot)$ [3, 8]. This leads to the Riemann-Hilbert problem to be discussed below.

⁹Typically, it could be a scale which dominates the process under consideration.

We now want to investigate to what extent the map $R : V \rightarrow V$ can be lifted to an automorphism $\Theta_R : H \rightarrow H$ of the Hopf algebra. We regard V as the space in which Feynman graphs evaluate by the Feynman rules, as discussed above. Let again the Feynman rules be implemented by ϕ . The map S_R is then a character constructed with the help of ϕ , so we should write $S_R \equiv S_R^\phi$ to be exact.

The question is if one can construct, for any R , an automorphism $\Theta_R : H \rightarrow H$ of the Hopf algebra such that one has

$$\bar{\phi} \equiv S_R^\phi \circ S = \phi \circ \Theta_R, \quad (24)$$

so that (using $S^2 = \text{id}$ which is true in any commutative Hopf algebra)

$$S_R \star \phi = \bar{\phi} \circ S \star \phi = \phi \circ \Theta_R \circ S \star \phi = \phi \circ [\Theta_R^{-1} \star \text{id}]. \quad (25)$$

The answer is affirmative [3]. Following [3] and the use of one-parameter group of automorphisms in the renormalization group [9] to be discussed below, we make the following *Ansatz* for Θ_R :

$$\Theta_R(\gamma) = e^{-\varepsilon \deg(\gamma) \rho_R(\gamma)}, \quad (26)$$

where, in the context of dimensional regularization or any other analytic regularization, $\rho_R(\gamma)$ will be a character evaluating in the ring of Taylor series in ε regular at $\varepsilon = 0$ and $\deg(\gamma) = n$ if γ has n loops.¹⁰ Then, one determines

$$\rho_R(\gamma) = \frac{-1}{\varepsilon \deg(\gamma)} \log \left(\frac{S_R^\phi \circ S(\gamma)}{\phi(\gamma)} \right), \quad (27)$$

so that indeed $\rho_R(\gamma)$ is free of poleterms, as one easily shows

$$\frac{S_R^\phi \circ S(\gamma)}{\phi(\gamma)} = 1 + \mathcal{O}(\varepsilon),$$

for arbitrary graphs γ . This gives a unifying approach to the treatment of renormalization schemes and changes between them.¹¹

4 The insertion operad of Feynman graphs

In this section, we want to describe an operad structure on Feynman graphs. This operad was implicitly present in many results in [5, 8, 9], and so it is worth

¹⁰It is convenient but not necessary to work with dimensional regularization here. In BPHZ, one could work for example with the ratio of Taylor series in external parameters.

¹¹From here, one can start considering categorical aspects of renormalization theory and in particular address the question posed in [2] if a modified coproduct $\Delta_R = (\Theta_R \otimes \text{id}) \circ \Delta$ is (weak-)coassociative in dependence of R , with first results upcoming in a recent thesis [24].

to describe it shortly at this stage, also with regard to the fact that it will prove to be a useful construct to investigate the number-theoretic aspects of Feynman graphs [13, 14] to be discussed below.

While the previous two sections discussed the process of disentangling a Feynman graph into subgraphs according to the presence of UV singularities, we now turn to the process of plugging graphs into each other. This will lead us in the next section to Lie algebras of Feynman graphs. Here, we want to study the most basic operation: plugging one graph γ_1 into another graph γ_2 . Typically, there are various places in γ_2 , provided by edges and vertices of γ_2 , which can be replaced by γ_1 . To obtain a sensible notion of this operation we should fulfill operad laws in this process. These operad laws can be described as follows. Operad laws are concerned with rules which should be fulfilled when we insert several times. First, assume we have graphs γ_1, γ_2 and want to plug both of them into different places of a graph γ . Then, the result should be independent of the order in which we do it. Next, when we plug γ_1 into γ_2 at some place, and insert the result into γ , the result should be the same as inserting γ_2 at the same place in γ , and then γ_1 into the corresponding relabelled place of γ_2 . Finally, the permutation of places should be compatible with the composition (see for example [25] for a formal definition of these requirements).

We only describe the operad in the context of massless ϕ^3 theory in six dimensions, the generalizations to more general cases are obvious and will be discussed elsewhere.

A Feynman graph provides vertices and edges connecting these vertices. The operad essentially consists of regarding these vertices and edges as places into which other graphs can be inserted. Naturally, a vertex correction can replace a vertex of a similar type, and a propagator-function can replace a line which represents a free propagator of a similar type. In massless ϕ^3 theory, we only have one type of lines and one type of vertices.

First, we note that the overall divergent Feynman graphs in this theory are given by 1PI graphs with two or three amputated external lines. Thus, vertices in the graphs are either internal three-point vertices, or two-point vertices resulting from the amputation of an external leg from a three-point vertex. Hence, self-energies can be described as graphs which precisely have two two-point vertices, while three-point graphs, –vertex corrections–, have precisely three two-point vertices. Propagator-functions then have two external edges.

When we want to replace an internal vertex, we just replace it by a vertex correction. When we want to replace an internal edge, a free propagator, we replace it by a propagator-function, as described by Figs.(13,14).

How many places are there? Let $\gamma(p_1, p_2)$ be a 1PI vertex function given by a three point graph γ , with l loops, which then provides $2l + 1$ vertices and $3l$ internal lines, hence $5l + 1$ places for insertion altogether. Let $\Pi(p)$ be a propagator function given by a (not necessarily one-particle irreducible) two-point graph Π with l loops, it then provides $2l$ vertices and $3l + 1$ lines, hence again $5l + 1$ places (we not necessarily have to label all edges and vertices, for

example dropping the label at an external edge of the propagator function takes into account quite naturally the fact that self-energies are proportional to an inverse propagator, and, in a massless theory, cancel one of the external lines).

We label all edges and vertices in arbitrary order, and the composition laws described in the figure captions of Figs.(13,14) fulfill the operad laws (the before-mentioned requirements are fulfilled), so that Figs.(13,14) define this operad by way of example.

So, with these rules for insertion (we also understand that insertion of a propagator-function at a vertex place or a vertex-function at an edge vanishes trivially by definition), one gets indeed an (partial) operad. Note further that insertion of a free propagator or vertex leaves the result unchanged.

One easily extends this construction to the case that one has vertices of other valencies and with different sorts of lines coming in.

This operad can be conveniently used to study the Lie algebraic structure of diagrams as well as for the investigation of number-theoretic aspects as we will see below. Also, the operad viewpoint is helpful in understanding the equivalence classes discussed in [2]. For example, the two graphs γ_1 and γ_2 of section 2.1 belong to the same equivalence class, $\gamma_1 \sim \gamma_2$, given by the parenthesized word $((\gamma), \circ)$, and are distinguished only by the place into which we insert γ . In general, two graphs are equivalent if one is obtained via a permutation of concatenation labels of the other, while maintaining the tree structure of its subdivergences: all Feynman graphs which represent the same rooted tree or parenthesized word can be obtained from each other by the change of labels of places where we insert the primitive graphs into each other.

Also, typical equations in field theory like Schwinger-Dyson equations are naturally formulated by this operad, using the fact that the sum over all diagrams can be written as a sum over all primitive ones into which all diagrams are plugged in all possible places. Details will be given in future work.

5 The Lie algebra structure

In [5, 8, 9] the reader finds various Lie algebra structures which appear in the dual of the Hopf algebra which is the universal enveloping algebra of a Lie algebra. Here, we describe the Lie algebra of Feynman graphs. There is also one for rooted trees, which can be found in [5].

Study of these Lie algebras is a very convenient way of understanding the structure of Feynman graphs. These Lie algebras play a crucial role when one wants to understand the connection between the group of diffeomorphisms of physical parameters like coupling constants with the group of characters of the Hopf algebra, to which we will turn in the next section.

It is also quite useful in determining the Hopf algebra structure of a chosen QFT correctly, because, once it is found, the corresponding enveloping algebra will be the dual of a commutative non-cocommutative Hopf algebra (by the

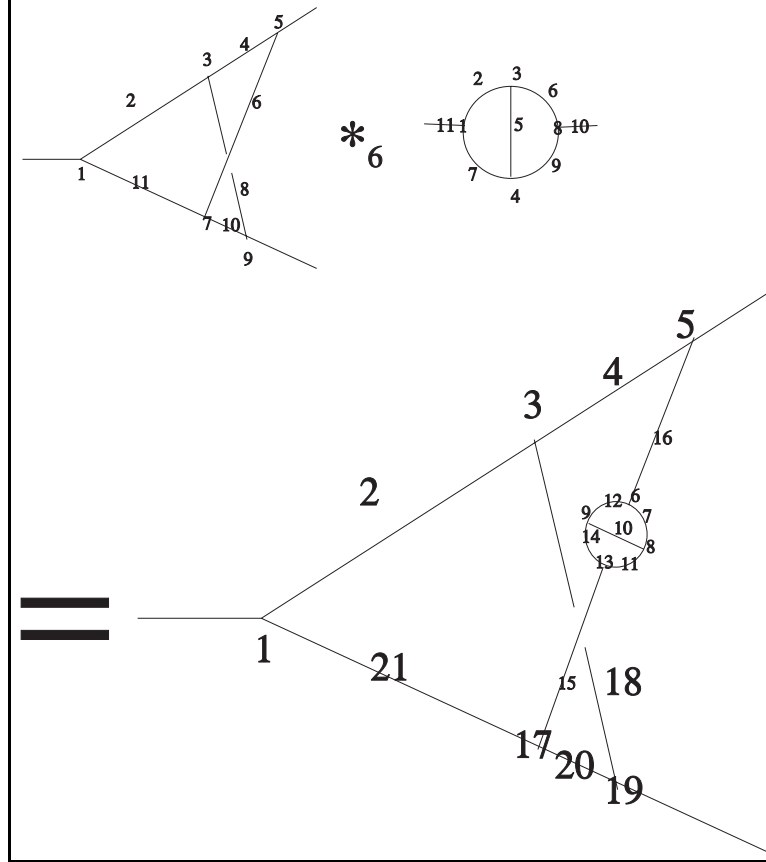


Figure 13: We consider a propagator-graph γ and a vertex-function γ , and as an example their concatenation $\gamma \star_6 \gamma$. The propagator function replaces the line with label 6 in the vertex-function. The propagator-function provides four vertices (labelled 1,3,4,8) and seven edges (labelled 2,5,6,7,9,10,11). Two of the edges, 10 and 11, are external. The vertex-function provides five vertices (labelled 1,3,5,7,9) and six edges (labelled 2,4,6,8,10,11). The vertices 1,5,9 are external, they connect to edges which are not part of the vertex function. We still indicated them by open-ended lines at those vertices, but one should regard vertices 1,5,9 as two-point vertices.

Note that each internal edge ends in two labelled vertices. We replace the edge labelled 6 by the propagator-function, connecting the external edges 10 and 11 of the latter to the vertices 5 and 7 of the vertex-function. We glue the edge with the lower label (10) to the vertex with the lower label (5). Relabelling is done in the obvious way: labels 1 to 5 in the vertex-function remain unchanged, the labels at the inserted propagator function become labels 6 to 16, and labels 7 to 11 become labels 17 to 21, increasing their labels by $4 + 7 - 1 = 10$.

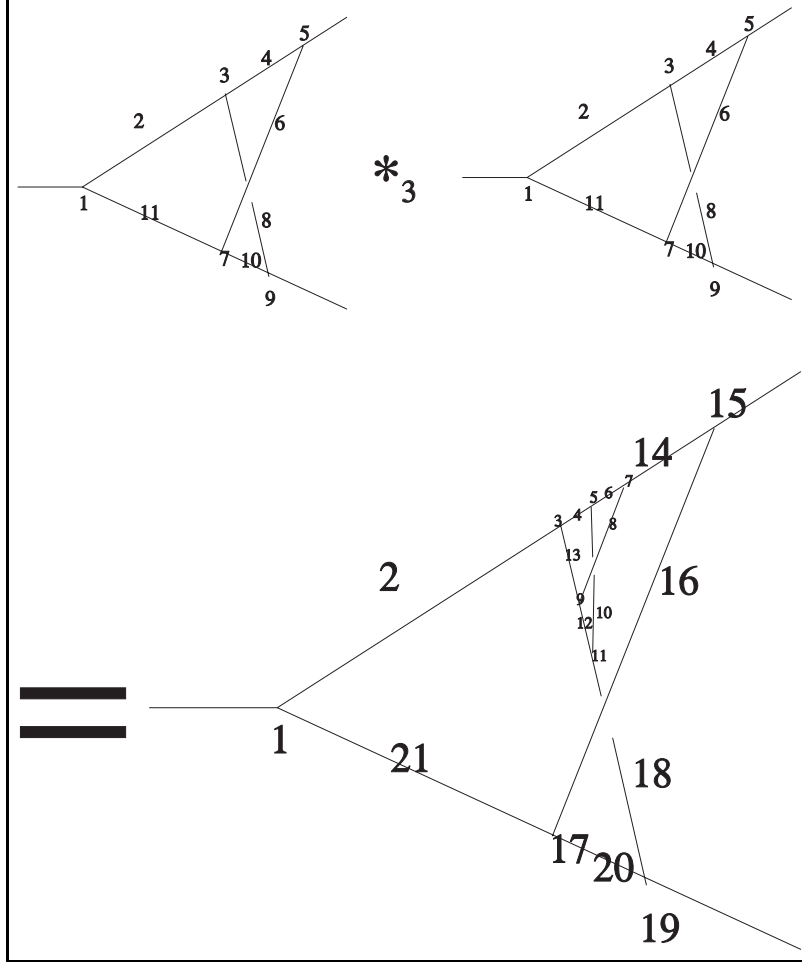


Figure 14: To explain the insertion of a vertex-function, we replace in this example vertex 3 of a vertex-function by the very same vertex-function, so we describe \cdot , \star_3 , \cdot . We do it by connecting edges 2,4,8 which are attached to vertex 3 to the three two-point vertices 1,5,9, respecting the order: edge 2 connects to vertex 1, edge 4 to vertex 5, edge 8 to vertex 9. Relabelling is done in the obvious way: labels 1 and 2 in the vertex-function remain unchanged, the labels at the inserted vertex-function become labels 3 to 13, and labels 4 to 11 become labels 14 to 21.

celebrated Milnor–Moore theorem [5, 8]) whose coproduct gives us the forests formulas of renormalization.¹² To find these Lie algebras, one defines a Lie-bracket of two 1PI graphs γ_1, γ_2 by plugging γ_1 into γ_2 in all possible ways and subtracts all ways of plugging γ_2 into γ_1 .

These Lie algebras all arise from a pre-Lie structure which we can describe in Fig.[15]. The operation of inserting one graph γ_1 in another graph γ_2 in all possible ways is a pre-Lie operation $\gamma_2 \star \gamma_1$, which means that it fulfills

$$\gamma_3 \star (\gamma_2 \star \gamma_1) - (\gamma_3 \star \gamma_2) \star \gamma_1 = \gamma_3 \star (\gamma_1 \star \gamma_2) - (\gamma_3 \star \gamma_1) \star \gamma_2.$$

Antisymmetrization then gives automatically a bracket $[\gamma_1, \gamma_2] = \gamma_1 \star \gamma_2 - \gamma_2 \star \gamma_1$, which fulfills the Jacobi identity. This operation of inserting one graph in another in all possible ways can obviously be written with the help of the operad structure of the previous section as a sum over all places where to insert (plus a sum over all permutations of the labels of identical external vertices of the graph which is to be inserted) and the operad laws then guarantee that the pre-Lie property is fulfilled, making use of the intimate connection between rooted trees, operads and pre-Lie algebras [28].

Once this Lie algebra is found, one knows that dually one obtains a commutative, non-cocommutative Hopf algebra which is the basis of the forest formulas of renormalization as discussed in the previous section.

It is not difficult to work out the corresponding pre-Lie structure for QED for example, and indeed, reading the graphs of Fig.(15) as QED graphs in the obvious possible manners only demands to cancel a few of the terms in that figure, because a photon propagator can only replace a photon line, and not a fermion line. Similarly, for any local QFT, one can determine the corresponding Hopf and Lie algebras, incorporating external structures whenever necessary as in [8].

The resulting Lie algebras of Feynman graphs play a fundamental role in understanding how the combinatorial properties of renormalization connect to the renormalization group, to the running of physical parameters. We now turn to study these results of [7, 8, 9].

6 The Birkhoff decomposition and the renormalization group

In [7, 8, 9] the reader finds an amazing connection between the Riemann–Hilbert problem and renormalization. This result was first announced in [7]. It is

¹²For example one easily determines the Lie algebra of QED, having one type of vertex connecting to two different type of lines for fermion and photon propagators. This then confirms the corresponding Hopf algebra structure of 1PI graphs to be commutative non-cocommutative. One-particle reducible graphs can be treated as in [16]. In the literature, there are other attempts to describe the renormalization of QED by binary rooted trees [26]. But the singularities of QED are stratified along diagonals as in any local QFT, and the rather artificial restriction to binary rooted trees ultimately runs into trouble [27].

$$\begin{aligned}
& \text{---}\bigcirc\text{---} * (\text{---}\triangleleft * \text{---}\bigcirc\text{---}) \\
& -(\text{---}\bigcirc\text{---} * \text{---}\triangleleft) * \text{---}\bigcirc\text{---} \\
& = \text{---}\bigcirc\text{---} * (\text{---}\triangleleft + \text{---}\triangleleft + \text{---}\triangleleft) \\
& \quad - (2 \text{---}\bigcirc\text{---}) * \text{---}\bigcirc\text{---} \\
& = \text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---} + 2 \text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---} \\
& \quad - 2(\text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---}) \\
& = -\text{---}\bigcirc\text{---} - \text{---}\bigcirc\text{---} - \text{---}\bigcirc\text{---} - \text{---}\bigcirc\text{---} \\
& = 2(\text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---}) \\
& \quad - (2 \text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---} + 2 \text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---}) \\
& = \text{---}\bigcirc\text{---} * (2 \text{---}\bigcirc\text{---}) \\
& \quad - (\text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---}) * \text{---}\triangleleft \\
& = \text{---}\bigcirc\text{---} * (\text{---}\bigcirc\text{---} * \text{---}\triangleleft) \\
& \quad - (\text{---}\bigcirc\text{---} * \text{---}\bigcirc\text{---}) * \text{---}\triangleleft
\end{aligned}$$

Figure 15: The (pre-)Lie algebra structure of Feynman graphs. The fact that the operation of plugging a graph into another one in all possible ways is pre-Lie is essentially due to the fact that the ways of plugging (in all possible ways) γ_1 into γ_2 , and the result into γ_3 , subtracted from the ways of plugging (in all possible ways) γ_1 into the result of plugging (in all possible ways) γ_2 into γ_3 is the sum over all possible ways to plug γ_1, γ_2 disjointly into γ_3 .

based on the use of a complex regularization parameter. Typically, dimensional regularization provides such a parameter as the deviation ε from the relevant integer dimension of spacetime, but for example analytic regularization would do as well.

With such a regularization parameter, the Feynman rules map a Feynman graph to a Laurent series with poles of finite order in this regularization parameter, hence the Feynman rules provide a character from the Hopf algebra of Feynman graphs to the ring of Laurent polynomials with poles of finite order in ε .

As mentioned before, the multiplicativity constraints [3, 7, 8]

$$R[xy] + R[x]R[y] = R[R[x]y] + R[xR[y]]$$

ensure that the corresponding counterterm map S_R is a character as well,

$$S_R[xy] = S_R[x]S_R[y], \quad \forall x, y \in H.$$

We now study how this set-up leads to the Riemann–Hilbert problem and the Birkhoff decomposition.

6.1 Minimal subtraction: the Birkhoff decomposition

To make contact with the Riemann–Hilbert problem, the crucial step is to recognize that, for $R = MS$ being chosen to be projection onto these poles of finite order (the minimal subtraction scheme MS), $\phi = S_{MS} \circ S \star [S_{MS} \star \phi]$ is a decomposition of the character ϕ into a part which is holomorphic at $\varepsilon = 0$: $S_{MS} \star \phi \equiv \phi_+$ is a character evaluating in the ring of functions holomorphic at $\varepsilon = 0$, while $S_{MS} \equiv \phi_-$ maps to polynomials in $1/\varepsilon$ without constant term, it delivers, when evaluated on Feynman graphs, the MS counterterms for those graphs. This corresponds to a Birkhoff decomposition $\phi = \phi_-^{-1} \phi_+$. For an introduction to the Riemann–Hilbert problem and the associated Birkhoff decomposition we refer the reader to [29]. Suffices it here to say that the Riemann–Hilbert problem is a type of inverse problem. For a given complex differential equation

$$y'(z) = A(z)y(z), \quad A(z) = \sum_i \frac{A_i}{z - z_i}$$

with given regular singularities z_i and matrices A_i , one can determine monodromy matrices M_i integrating around curves encircling the singularities. The inverse problem, finding the differential equation from knowledge of the singular places and monodromy matrices, is the Riemann–Hilbert problem. A crucial role in its solution plays the Birkhoff decomposition: for a closed curve C in the Riemann sphere, and a matrix-valued loop $\gamma : z \rightarrow \gamma(z)$ well-defined on C , decompose it into parts γ_{\pm} well-defined in the interior/exterior of C .

Thus, renormalization in the MS scheme can be summarized in one sentence: with the character ϕ given by the Feynman rules in a suitable regularization

scheme and well-defined on any small curve around $\varepsilon = 0$, find the Birkhoff decomposition $\phi_+(\varepsilon) = \phi_- \phi$, where now and in the following the product in expressions like $\phi_- \phi$ is meant to be just the convolution product $\phi_- \star \phi$ of characters used before.

The unrenormalized analytic expression for a graph γ is then $\phi[\gamma](\varepsilon)$, the MS-counterterm is $S_{MS}(\gamma) \equiv \phi_-[\gamma](\varepsilon)$ and the renormalized expression is the evaluation $\phi_+[\gamma](0)$. Once more, note that the whole Hopf algebra structure of Feynman graphs is present in this group: the group law demands the application of the coproduct, $\phi_+ = \phi_- \phi \equiv S_{MS} \star \phi$.

The transition from here to other renormalization schemes can be achieved in the group of characters in accordance with our previous considerations in section 3.

But still, one might wonder what a huge group this group of characters really is. What one confronts in QFT is the group of diffeomorphisms of physical parameter: low and behold, changes of scales and renormalization schemes are just such (formal) diffeomorphisms. So, for the case of a massless theory with one coupling constant g , for example, this just boils down to formal diffeomorphisms of the form

$$g \rightarrow \psi(g) = g + c_2 g^2 + \dots$$

The group of one-dimensional diffeomorphisms of this form looks much more manageable than the group of characters of the Hopf algebras of Feynman graphs of this theory.

Thus, it would be very nice if the whole Birkhoff decomposition could be obtained at the level of diffeomorphisms of the coupling constants, and this is what was achieved in [9].

6.2 The β -function

Following [8] in the above we have seen that perturbative renormalization is a special case of a general mathematical procedure of extraction of finite values based on the Riemann-Hilbert problem. The characters of the Hopf algebra of Feynman graphs form a group whose concatenation, unit and inverse are given by the coproduct, the counit and the antipode. So we can associate to any given renormalizable quantum field theory an (infinite dimensional) complex Lie group G of characters of its Hopf algebra H of Feynman graphs. Passing from the unrenormalized theory to the renormalized one corresponds to the replacement of the loop $\varepsilon \rightarrow \gamma(\varepsilon) \in G$ (obtained by restricting the character ϕ to an arbitrarily chosen curve C around $\varepsilon = 0$) of elements of G obtained from dimensional regularization (still, $\varepsilon \neq 0$ is the deviation from the integer dimension of space-time) by the value $\gamma_+(\varepsilon)$ of its Birkhoff decomposition, $\gamma(\varepsilon) = \gamma_-(\varepsilon)^{-1} \gamma_+(\varepsilon)$.

In [9] it was shown how to use the very concepts of a Hopf and Lie algebra of graphs to lift the usual concepts of the β -function and renormalization group

from the space of coupling constants of the theory to the complex Lie group G . We now exhibit these results.

The original loop $\varepsilon \rightarrow \gamma(\varepsilon)$ not only depends upon the parameters of the theory but also on the additional *unit of mass* μ , –the 't Hooft mass in dimensional regularization–, required by dimensional analysis.

But although the loop $\gamma(\varepsilon)$ does depend on the additional parameter μ ,

$$\mu \rightarrow \gamma(\varepsilon; \mu) ,$$

the negative part $\gamma_{\mu-}$ in the Birkhoff decomposition, the character delivering the MS counterterms,

$$\gamma(\varepsilon; \mu) = \gamma_{-}(\varepsilon; \mu)^{-1} \gamma_{+}(\varepsilon; \mu)$$

is actually independent of μ ,

$$\frac{\partial}{\partial \mu} \gamma_{-}(\varepsilon; \mu) = 0 . \quad (28)$$

This is a remnant of the fact that our Hopf algebra is constructed so as to achieve local counterterms: ϕ is a character which can be easily shown to be a series in $\log(q^2/\mu^2)$ so that a remaining μ^2 dependence in MS counterterms would be accompanied by a remaining q^2 dependence, and would hence violate locality.¹³

The Lie group G turns out to be graded, with grading,

$$\theta_{\rho} \in \text{Aut } G , \quad \rho \in \mathbb{R} ,$$

inherited from the grading of the Hopf algebra H of Feynman graphs given by the loop number,

$$\deg(,) = \text{loop number of } , \quad (29)$$

for any 1PI graph $, ,$ so that $\theta_{\rho}(,) = e^{\rho \deg(\Gamma)}$,¹⁴

This leads to

$$\gamma(\varepsilon; e^{\rho} \mu) = \theta_{\rho \varepsilon}(\gamma(\varepsilon; \mu)) \quad \forall \rho \in \mathbb{R} ,$$

¹³A similar argument applies when the Feynman rules provide a character parametrized by several scales. Again, by a group action which is a finite rerenormalization, we can reduce the unrenormalized theory to a dependence on a single scale. This reduction can constrain the renormalization group flow to a submanifold though, in which case an explicit group action is needed to switch from mass-independent to mass-dependent renormalization group functions, as it is well-known [30].

¹⁴Here ρ is to be regarded as a constant. If we promote it to a character evaluating in the ring of functions holomorphic at $\varepsilon = 0$ we obtain the automorphisms used in section 3 to lift the renormalization map R to automorphisms of the Hopf algebra. Note that a constant ρ is sufficient to describe momentum schemes for example, using that one only has to use $\rho = \varepsilon \log(\mu^2/q^2)$ to compensate for the canonical q^2 -dependence [2, 3, 16, 10].

so that the loops $\gamma(\mu)$ associated to the unrenormalized theory have the property that the negative part of their Birkhoff decomposition is unaltered by the operation,

$$\gamma(\varepsilon) \rightarrow \theta_{\rho\varepsilon}(\gamma(\varepsilon)) :$$

if we replace $\gamma(\varepsilon)$ by $\theta_{\rho\varepsilon}(\gamma(\varepsilon))$ we do not change the negative part of its Birkhoff decomposition. A complete characterization of the loops $\gamma(\varepsilon) \in G$ fulfilling this invariance can be found in [9]. This characterization only involves the negative part $\gamma_-(\varepsilon)$ of their Birkhoff decomposition which by hypothesis fulfills,

$$\gamma_-(\varepsilon) \theta_{\rho\varepsilon}(\gamma_-(\varepsilon))^{-1} \text{ is convergent for } \varepsilon \rightarrow 0 . \quad (30)$$

It is then easy to see that this defines in the limit $\varepsilon \rightarrow 0$ a one parameter subgroup,

$$F_\rho \in G , \quad \rho \in \mathbb{R} . \quad (31)$$

Now, the role of the β -function is revealed: the generator $\beta := \left(\frac{\partial}{\partial \rho} F_\rho \right)_{\rho=0}$ of this one parameter group is related to the *residue* of the loop γ

$$\text{Res}_{\varepsilon=0} \gamma = - \left(\frac{\partial}{\partial u} \gamma_- \left(\frac{1}{u} \right) \right)_{u=0} \quad (32)$$

by the simple equation,

$$\beta = Y \text{ Res } \gamma , \quad (33)$$

where $Y = \left(\frac{\partial}{\partial \rho} \theta_\rho \right)_{\rho=0}$ is the grading. In a moment, we will see how this generator β relates to the common β -function of physics.

All this is a simple consequence of the set-up described so far and is worked out in detail in [9] (essentially, at the moment we quote a summary of the results of that paper), while the central result of [9] gives $\gamma_-(\varepsilon)$ in closed form as a function of β . Let us use an additional generator in the Lie algebra of G (i.e. primitive elements of H^*) implementing the grading such that $[Z_0, X] = Y(X) \forall X \in \text{Lie } G$. Then, the loop $\gamma_-(\varepsilon)$ corresponding to the MS counterterm evaluated on any close curve around $\varepsilon = 0$ can be written by a scattering type formula for $\gamma_-(\varepsilon)$ as

$$\gamma_-(\varepsilon) = \lim_{t \rightarrow \infty} e^{-t(\frac{\beta}{\varepsilon} + Z_0)} e^{tZ_0} . \quad (34)$$

Both factors in the right hand side belong to the semi-direct product,

$$\tilde{G} = G \rtimes_{\theta} \mathbb{R}$$

of the group G by the grading, but their product belongs to the group G .

As a consequence the higher pole structure of the divergences is uniquely determined by the residue and this gives a strong form of the t'Hooft relations, which come indeed as an immediate corollary.¹⁵

The most fundamental result of [9] is obtained though when considering two competing Hopf algebra structures: diffeomorphisms of physical parameters carry, being formal diffeomorphisms, with them the Hopf algebra structure of such diffeomorphisms. This structure was recognized for the first time by Alain Connes and Henri Moscovici in [31]. On the other hand, a variation of physical parameters induced by a variation of scales is a renormalization, which directly leads to the Hopf algebra of Feynman graphs. Let us first describe the Hopf algebra structure of the composition of diffeomorphisms in a fairly elementary way, while mathematical detail can be found in [31].

Assume you have formal diffeomorphisms ϕ, ψ in a single variable

$$x \rightarrow \phi(x) = x + \sum_{k \geq 1} c_k^\phi x^k, \quad (35)$$

and similarly for ψ . How do you compute the Taylor coefficients $c_k^{\phi \circ \psi}$ for the composition $\phi \circ \psi$ from the knowledge of the Taylor coefficients c_k^ϕ, c_k^ψ ? It turns out that it is best to consider the Taylor coefficients

$$\delta_k^\phi = \log(\phi'(x))^{(k)}(0) \quad (36)$$

instead, which are as good to recover ϕ as the usual Taylor coefficients. The answer lies then in a Hopf algebra structure:

$$\delta_k^{\phi \circ \psi} = m \circ (\tilde{\psi} \otimes \tilde{\phi}) \circ \Delta_{CM}(\delta_k),$$

where $\tilde{\phi}, \tilde{\psi}$ are characters on a certain Hopf algebra H_{CM} (with coproduct Δ_{CM}) so that $\tilde{\phi}(\prod_i \delta_i) = \prod_i \delta_i^\phi$. Thus one finds a Hopf algebra with abstract generators δ_n such that it introduces a convolution product on characters evaluating to the Taylor coefficients $\delta_n^\phi, \delta_n^\psi$, such that the natural group structure of these characters agrees with the diffeomorphism group.

It turns out that this Hopf algebra of Connes and Moscovici is intimately related to rooted trees in its own right [5], signalled by the fact that it is linear in generators on the rhs, as are the coproducts of rooted trees and graphs.¹⁶

¹⁵The explicit formulas in [9] allow to find the combinations of primitive graphs into which higher order poles resolve. The weights are essentially given by iterated integrals which produce coefficients which generalize the tree-factorials obtained for the undecorated Hopf algebra in [3, 16, 10]. Iterated application of this formula allows to express inversely the first-order poles contributing to the β -function as polynomials in Feynman graphs free of higher-order poles.

¹⁶Taking the δ_n as naturally grown linear combination of rooted trees imbeds the commutative part of the Connes-Moscovici Hopf algebra in the Hopf algebra of rooted trees, which on the other hand allows for extensions similar to the ones needed by Connes and Moscovici. Details are in [5].

This initiated the collaboration of Alain Connes and the author, when, in a lucky accident, we both stumbled over similar Hopf algebras at about the same time.

Now, following [9], let us specialize to the massless case. Then the formula for the bare coupling constant,

$$g_0 = g Z_1 Z_3^{-3/2} \quad (37)$$

(where both $g Z_1 = g + \delta g$ and the field strength renormalization constant Z_3 are thought of as power series (in g) of elements of the Hopf algebra H) does define a Hopf algebra homomorphism,

$$H_{CM} \xrightarrow{g_0} H ,$$

from the Hopf algebra H_{CM} of coordinates on the group of formal diffeomorphisms of \mathbb{C} (ie such that $\varphi(0) = 0$, $\varphi'(0) = \text{id}$ as in Eq.(35)) to the Hopf algebra H of the massless theory.¹⁷ Having this Hopf algebra homomorphism from H_{CM} to H , dually one gets a transposed group homomorphism ρ , a homomorphism from the huge group of characters of the Hopf algebra to the group of diffeomorphism of physical parameters [9]. We finally recover the usual β -function: the image by ρ of the previously introduced generator $\beta = Y \text{ Res } \gamma$ is then the usual β -function of the coupling constant g . While this might sound rather abstract, it can be easily translated into the standard notions of renormalization theory (see, for example, [32]).

While in [9] the physical parameter under consideration was a single coupling, similar considerations apply to other physical parameters which run under the renormalization group, making use of the Hopf algebraic description of composition of diffeomorphisms in general.

As a corollary of the construction of ρ one gets an *action* by (formal) diffeomorphisms of the group G on the space X of (dimensionless) coupling constants of the theory. One can then in particular formulate the Birkhoff decomposition *directly* in the group $\text{Diff}(X)$ of formal diffeomorphisms of the space of coupling constants.

The unrenormalized theory delivers a loop

$$\delta(\varepsilon) \in \text{Diff}(X) , \quad \varepsilon \neq 0 ,$$

whose value at $\varepsilon \neq 0$ is simply the unrenormalized effective coupling constant. The Birkhoff decomposition $\delta(\varepsilon) = \delta_+(\varepsilon) \delta_-(\varepsilon)^{-1}$ of this loop gives directly

$$\delta_-(\varepsilon) = \text{bare coupling constant}$$

¹⁷We restrict ourselves to the massless theory so that we can deal with one-dimensional diffeomorphisms. We can regard a mass as a further coupling constant of a two-point vertex which leads to formal diffeomorphisms of higher dimensional spaces.

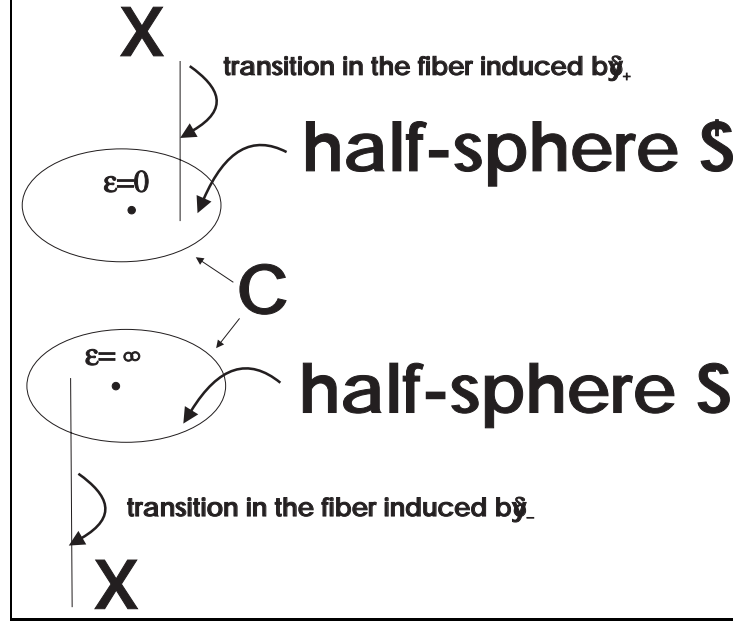


Figure 16: The geometric picture of [9] allows for the construction of a complex bundle, $P = (S^+ \times X) \cup_{\delta} (S^- \times X)$ over the sphere $S = P_1(\mathbb{C}) = S^+ \cup S^-$, and with fiber X , $X \longrightarrow P \xrightarrow{\pi} S$, where X is a complex manifold of physical parameters. The transition in this fiber are diffeomorphisms. $\delta(\varepsilon)$ delivers a diffeomorphism of X for any $\varepsilon \in C$, where C is the boundary of the two half-spheres S^+, S^- . It extends to the interiors of the half-spheres via its Birkhoff decomposition. The meaning of this Birkhoff decomposition, $\delta(\varepsilon) = \delta_+(\varepsilon) \delta_-(\varepsilon)^{-1}$ is then exactly captured by an isomorphism of the bundle P with the trivial bundle, $S \times X$. Note that $\delta_-(\infty)$ is well-defined due to the fact that S_{MS} has no constant term in ε , which characterizes a minimal subtraction scheme.

and

$$\delta_+(\varepsilon) = \text{renormalized effective coupling constant.}$$

This result is now stated in a manner independent of our group G or the Hopf algebra H , its proof makes heavy use of these ingredients though.

Finally, the Birkhoff decomposition of a loop, $\delta(\varepsilon) \in \text{Diff}(X)$ admits a beautiful geometric interpretation [9], described in Fig.(16).

6.3 An example

In [9] the reader can find explicit computational examples up to the three-loop level, and a complete proof to all loop orders, for the group and Hopf algebra homomorphisms described above. We only want to check the Hopf algebra

homomorphism $H_{CM} \rightarrow H$ up to two loops here. We regard g_0 as a series in a variable x (which can be thought of as a physical coupling) up to order x^4 , making use of $g_0 = xZ_1Z_3^{-3/2}$ and the expression of the Z -factors in terms of 1PI Feynman graphs. The challenge is then to confirm that the coordinates $\delta_n^{g_0}$, implicitly defined by [31]

$$\log [g_0(x)]^{(n)},$$

as expected from Eq.(36), commute with the Hopf algebra homomorphism: calculating the coproduct Δ_{CM} of δ_n and expressing the result in terms of Feynman graphs with the help of the character corresponding to g_0 , $\tilde{g}_0(\delta_n) = \delta_n^{g_0}$, must equal the application of the coproduct Δ applied to $\delta_n^{g_0}$.

We write $g_0 = xZ_1Z_3^{-3/2}$,

$$Z_1 = 1 + \sum_{k=1}^{\infty} z_{1,2k} x^{2k},$$

$$Z_3 = 1 - \sum_{k=1}^{\infty} z_{3,2k} x^{2k},$$

and

$$Z_g = Z_1Z_3^{-3/2}, \quad z_{i,2k} \in \mathcal{H}_c, \quad i = 1, 3,$$

as formal series in x^2 . Using

$$\log \left(\frac{\partial}{\partial x} x Z_g \right) = \sum_{k=1}^{\infty} \frac{\delta_{2k}^{g_0}}{(2k)!} x^{2k},$$

which defines $\delta_{2k}^{g_0}$, we find

$$\begin{aligned} \frac{1}{2!} \delta_2^{g_0} \equiv \tilde{\delta}_2^{g_0} &= 3z_{1,2} + \frac{9}{2} z_{3,2}, \\ \frac{1}{4!} \delta_4^{g_0} \equiv \tilde{\delta}_4^{g_0} &= 5[z_{1,4} + \frac{3}{2} z_{3,4}] - \frac{9}{2} z_{1,2}^2 - 6z_{1,2} z_{3,2} - \frac{3}{4} z_{3,2}^2. \end{aligned} \quad (38)$$

The algebra homomorphism $H_{CM} \rightarrow H$ is effected by expressing the $z_{i,2k}$ in Feynman graphs, with 1PI graphs with three external legs contributing to Z_1 , and 1PI graphs with two external legs, self-energies, contributing to Z_3 .

Explicitly, we have

$$\begin{aligned} z_{1,2} &= \triangleleft, \\ z_{3,2} &= \frac{1}{2} \text{---}\bigcirc\text{---}, \\ z_{1,4} &= \triangleleft + \triangleleft + \triangleleft + \frac{1}{2} \left[\triangleleft + \triangleleft + \triangleleft \right] + \frac{1}{2} \triangleleft, \\ z_{3,4} &= \frac{1}{2} \left[\text{---}\bigcirc\text{---} + \text{---}\bigoplus\text{---} \right]. \end{aligned}$$

On the level of diffeomorphisms, we have the coproduct

$$\Delta_{CM}[\delta_4] = \delta_4 \otimes 1 + 1 \otimes \delta_4 + 4\delta_2 \otimes \delta_2, \quad (39)$$

where we skip odd gradings (in ϕ^3 theory, adding a loop order increases the order in the coupling by g^2).

We have to check that the coproduct Δ of Feynman graphs reproduces these results.

Applying Δ to the rhs of (39) gives, using the expressions for $z_{i,k}$ in terms of Feynman graphs,

$$\begin{aligned} \Delta(\tilde{\delta}_4) = & 6 \triangleleft \otimes \triangleleft + \frac{9}{2} \left[\triangleleft \otimes \bigcirc + \bigcirc \otimes \triangleleft \right] \\ & + \frac{27}{8} \bigcirc \otimes \bigcirc + \tilde{\delta}_4 \otimes 1 + 1 \otimes \tilde{\delta}_4. \end{aligned}$$

This has to be compared with $\tilde{\delta}_4 \otimes 1 + 1 \otimes \tilde{\delta}_4 + \frac{2!2!}{4!} 4\tilde{\delta}_2 \otimes \tilde{\delta}_2$, which matches nicely, as

$$\begin{aligned} \tilde{\delta}_2 \otimes \tilde{\delta}_2 = & 9 \triangleleft \otimes \triangleleft + \frac{27}{4} \left[\triangleleft \otimes \bigcirc + \bigcirc \otimes \triangleleft \right] \\ & + \frac{81}{16} \bigcirc \otimes \bigcirc. \end{aligned}$$

7 Conclusions and Outlook

In this final section we mainly want to comment on some more future lines of investigation, which in part are already work in progress. We start with the connection between Feynman diagrams and the numbers which we see in their coefficients of ultraviolet divergence, which is a rich source of structure [15].

7.1 Numbers and Feynman diagrams

There is an enormous amount of interesting number theory in Feynman diagrams [33, 34, 15]. In particular, the primitive elements in the Hopf algebra, those graphs which have no subdivergences and provide a renormalization scheme independent coefficient of ultraviolet divergence, show remarkable and hard to explain patterns. These coefficients evaluate in Euler–Zagier sums (generalized polylogs evaluated at (suitable roots of) unity so that they generalize multiple zeta values (MZVs) [15, 33, 34]), numbers which have remarkably fascinating algebraic structure [35, 36, 37, 38].

These algebraic structures are believed to be governed by shuffle algebras, and by the much more elusive Grothendieck–Teichmüller group (see, for example, [39] for an introduction to the Grothendieck–Teichmüller group which is close in spirit to the consideration of short-distance singularities).

The coefficients of UV-divergence in Feynman diagrams typically evaluate, up to the six loop level, in terms of these Euler–Zagier sums, but the question if this will always be so remains open in light of the failure to identify all these coefficients in this number class at the seven loop level [33, 34, 15]. While the embarrassingly successful heuristic approach summarized in [15], providing a knot-to-number dictionary for those numbers, only emphasizes the need for a more thorough understanding, the algebraic structures in Feynman graphs hopefully lead to such an understanding in the future. It is already remarkable that shuffle products can be detected in Feynman graphs [13], but there are hints for much more structure [14].

But while the existence of shuffle algebras in Feynman graphs can essentially be straightforwardly addressed due to the fact that a shuffle algebra makes use of the B_+ , B_- operators in a natural way [13], these remaining algebraic relations between Feynman graphs will be harder to address.¹⁸ But the very fact that Feynman graphs realize their short-distance singularities in tree-like hierarchies suggests that they can be understood along lines similar to what is known for Euler–Zagier sums.

In particular, Feynman graphs whose subdivergences realize the same rooted tree but with subgraphs inserted at different internal lines provide remarkable number-theoretic features [40]. As mentioned before, in the operad picture, such differences are given by permutations $\sigma(i) = j$ of places i at which we compose:

$$, \circ_i \gamma \rightarrow , \circ_{\sigma(i)} \gamma.$$

Note that, if we let U be the difference of the two expressions, we get a primitive element in the Hopf algebra (if the two graphs $,$ and γ are both primitive), $\Delta(U) = U \otimes 1 + 1 \otimes U$.

Quite often, one finds that these differences are even finite, which means that the coefficients of ultraviolet divergence are the same and drop out in the difference: short distance singularities are invariant under the above permutations. Fig.(17) gives an example of such an invariance observed in [40]. We insert a one-loop bubble at different places i, j in the graph. We do not have to worry that in one case it is a one-loop fermion self-energy, in the other case a one-loop boson self-energy. In massless Yukawa theory, they both evaluate to the same analytic expression. This makes it very easy to study the effect of a subdivergence being inserted at different places in a larger graph. In this four-loop example, the difference becomes a primitive element and hence delivers only a first order pole $\sim \zeta(3)/\varepsilon$, signalling the difference in topology between

¹⁸But note that these shuffle algebras and shuffle identities only hold for the coefficients of ultraviolet singularity: they hold up to finite parts, up to finite renormalizations that is.

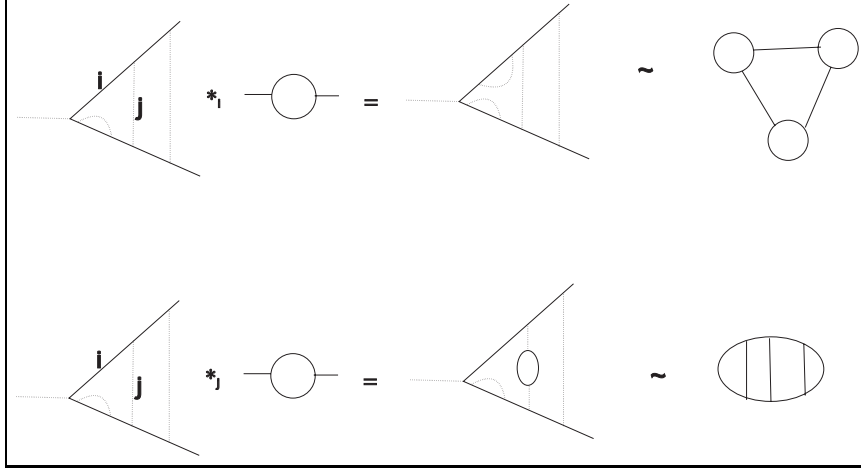


Figure 17: These two Feynman graphs (with their distinct topologies indicated on the rhs of each: the topology of the upper graph is that of disjoint one-loop insertions, the lower is a ladder topology) in massless Yukawa theory have a remarkable relation: their difference is a primitive Hopf algebra element. When evaluating the character S_{MS} on both, one finds a Laurent series with poles of fourth order from both of them. In the difference, the highest three-pole terms drop out, and the remaining term is $\sim \zeta(3)/\varepsilon$. Similar phenomena happen at higher loop orders [40]: higher poleterms are invariants under the permutation of places where we insert subgraphs.

the two diagrams [15]. The ladder diagram evaluates to rational coefficients in the poleterms of its MS counterterm, while the other diagram has the same rational part, but also has $\zeta(3)$ in the $1/\varepsilon$ pole. In the difference, only this first order pole $\sim \zeta(3)/\varepsilon$ remains.

Comparing the two three-loop subgraphs of each diagram, one finds their difference to be finite and $\sim \zeta(3)$, so that the three coefficients $\sum_{i=1}^3 c_i/\varepsilon^i$ are invariant under an exchange of the place where we insert the subgraph: the morphism sending one graph to the other, and thus sending one configuration of internal vertices with its characteristic short-distance singularities to another, is a finite one. Similar observation hold for higher loop orders [40].

A systematic understanding of such phenomena, and a possible relation to finite-type invariants, seems crucial to understand the algebraic relations in Feynman graphs completely. Ultimately, one hopes for a geometric understanding of the analytic challenge posed by Feynman diagrams. Meanwhile, similar relations have been observed in QED [41].

A requirement on the way to such an understanding is the question how in the geometric picture of Fig.(16) one can relate an infinitesimal variation in the base space to a variation in the fiber, ie the quest for a connection?

For the δ_- part of the Birkhoff decomposition, this leads to an investigation

as to how a derivative with respect to the regularization parameter ε is related to the insertion of a further graph. First results at low loop orders to be discussed elsewhere indicate that this is a source for relations between the coefficients of ultraviolet divergence similar but not quite like the four-term relations discussed in the study of finite-type invariants [15]. This is not impossible: while all higher poleterms are fixed in terms of the residue by the scattering type formula Eq.(34) of the previous section, this formula can by its very nature not deliver relations between residues of graphs.

7.2 Gauge symmetries

Clearly, one of the most urgent and fascinating questions is the role of symmetries in quantum field theories. Having, with the Hopf algebra structures reported here, discovered such a wonderful machinery which encapsulates the quest for locality, one should expect interesting structure when considering local gauge symmetries, in particular also with respect to the role which foliations play naturally in noncommutative geometry [31, 42].

There are many aspects which can hopefully be addressed in the near future.

- To what extent can Ward- and Slavnov-Taylor identities be incorporated in this picture? Do these identities form something like an ideal in the algebra of graphs? Note that the language of external structures allows nicely to formulate concepts like the longitudinal and transversal part of a vertex-correction for example, and is hence well-adopted to address such questions.
- Has BRST cohomology a natural formulation in this context?
- Gauge theories provide number-theoretical miracles in abundance, with the most significant observation being Jon Rosner's observation [43] of the vanishing of $\zeta(3)$ from the β -function of quenched QED. While this can be understood heuristically [44, 15], eventually the role between internal symmetries and number-theoretic properties must be properly understood.

For the practitioner of quantum field theory, the real challenge lies in the treatment of the perturbative expansion in circumstances when there is no regularization available which preserves the symmetries of the initial theory. A notorious and famous problem at hand is the γ_5 problem in dimensional regularization [45]. In realistic circumstances like the Standard Model this already demands a formidable effort at the one-loop level if one uses a calculational scheme which violates the BRS symmetry even in the absence of anomalies (see [46] for such an example), which then is an unavoidable effort dictated by the demand to restore the BRS symmetry using the quantum action principle. There is one obvious useful role for the Hopf algebra: the analysis at the one-loop level would in many ways not change when extended to any other primitive element of the

Hopf algebra, which, being primitive, all share with the one-loop graphs that they have no subdivergences. From there, the Hopf algebra structure governs the iteration of graphs into each other.

But then, the prominent role and natural role which field-theoretic ingredients like the Dirac propagator and γ_5 itself, a volume form on four-dimensional space essentially, play in non-commutative geometry [6, 42], gives hope for a more profound understanding of this problem in the future.

7.3 The exact renormalization group and the non-perturbative regime

Ultimately, the renormalization group is a non-perturbative object, and can indeed be addressed without necessarily making use of the usual concepts of graph-theoretic expansions [47, 48]. This is nicely reflected by the fact that the transition from the perturbative to the non-perturbative just amounts, in the picture outlined here, to a Birkhoff decomposition of an actual instead of a formal diffeomorphism. Integrating out high frequency modes in the functional integral step-by-step produces a sequence of diffeomorphisms of the correlation function under consideration.¹⁹

The Hopf algebra of rooted trees, thanks to its universality, provides the relevant backbone in any case, and indeed rooted trees underly any iterative equation, like, for example, the Wilson equation

$$\frac{\partial S_\lambda}{\partial \lambda} = \mathcal{F}(S_\lambda),$$

for some action parametrized by some cut-off λ and some suitable functional \mathcal{F} . Integrating this functional \mathcal{F} now plays the role of the operator B_+ in the universal setting of the Hopf algebra of rooted trees [5]. Rooted trees are deeply built into solutions of (integro-) differential equations [49, 50]. It is no miracle then that on the other hand one finds that the understanding of the Hopf- and Lie algebras of Feynman graphs not only enables high-loop order calculations [10, 11, 51] which allow to analyze Padé-Borel resummations [11, 51, 52] but also allows to find exact non-perturbative solutions in new problems. A first result can be found in [51].

7.4 Further aspects

Combinatorially, rooted trees are very fundamental objects, and their Hopf and Lie algebra structure underlies not only the combinatorial process of renormalization, but can hopefully be used in the future in other expansions in perturbation theory, starting from a disentanglement of infrared divergent sectors

¹⁹The fact, emphasized by Polchinski [47], that in such an approach one does not see the graph-theoretical notions emphasized in textbook approaches to renormalization theory is a mere reflection of the fact that one can formulate the Birkhoff decomposition directly on the level of diffeomorphisms of physical observables [9], as exhibited in the previous section.

[53] to more general applications in asymptotic expansions [54]. Its universal nature already allowed to use it in a straightforward formulation of block spin transformations, coarse graining and the renormalization of spin networks [55]. Eventually, one hopes, this basic universal combinatorial structure finds its way into other approaches to QFT, from the constructive approach [56] which in its nature is very tree-like from a start [57], to the algebraic school [58, 59], which all have to handle the basic combinatorial step that we can address a problem only after we addressed its subproblems.²⁰ Note also that applications of forest formulas in the context of noncommutative field theory and string field theory (see [62] for a detailed graphical analysis) naturally change the criteria for the subgraphs γ over which a sum

$$\Delta(\cdot, \cdot) = \cdot \otimes 1 + 1 \otimes \cdot + \sum_{\gamma} \gamma \otimes \cdot / \gamma$$

runs, while the results in [4] underline that a Hopf algebra structure can still be established when we vary these criteria.

There is no space here to comment in detail on some other mathematical developments which are related to the discovery of the Hopf algebra structure of renormalization. We can only address the interested reader to [63, 64, 65, 28]. But note that such mathematical investigations are often very useful for a practitioner of QFT: clearly, the classification of all primitive Hopf algebra elements is of importance even for the case of the undecorated Hopf algebra of rooted trees, and leads for example to the notion of a bigrading which characterizes potential higher divergences algebraically [12, 65].

7.5 Conclusions

Rooted trees and Feynman graphs are familiar objects for anybody working on the perturbative expansion of a functional integral, and as familiar are forest formulas and the Bogoliubov recursion.

What is new is that there is a universal Hopf algebra on rooted trees, devoted to the problem of singularities along diagonals in configuration spaces and providing a principle of multiplicative subtraction, which reproduces just these recursions and forest formulas. That Feynman graphs, with all their external structure, form a Lie algebra is a very nice consequence which hopefully gives a new and strong handle for the understanding of QFT in the future. The consequences of the connection to the Riemann–Hilbert problem and the Birkhoff decomposition of diffeomorphisms, the connection between short-distance singularities in perturbation theory and polylogarithms, all this indicates what a rich source of mathematical structure and beauty is imposed on a quantum field theory by its infinities.

²⁰The universality of the Hopf algebra can be used to describe effective actions in a unifying manner, which was indeed one of the main points of [8, 9], while the connection to integrable models promoted in [60, 61] can hopefully be substantiated further in the future.

Acknowledgments

A large body of the work presented here was done in past and ongoing collaborations with David Broadhurst and Alain Connes. Helpful discussions with Jim Stasheff on operads are gratefully acknowledged. This work was done in part for the Clay Mathematics Institute. Also, the author thanks the DFG for a Heisenberg fellowship.

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